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Research Article

Asymptotic Properties of the Semigroup Generated by a Continuous Interval Map

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Abstract: The article's purpose is twofold. First, we wish to draw attention to the insufficiently known field of continuous-time difference equations. These equations are paradigmatic for modeling complexity and chaos. Even the simplest equation $x(t+1)=f(x(t)),\ t>0$, easily leads to complex dynamics, its solutions are perfectly suited to simulate strong nonlinear phenomena such as large-to-small cascades of structures, intermixing, formation of fractals, etc. Second, in the main body of the article we present a small but very important part of the theory behind the above equation marked by (*). Just as the discrete-time analog of this equation induces the one-dimensional dynamical system $z\mapsto f(z)$ on some interval I, so the equation (*) induces the infinite-dimensional dynamical system $\varphi(z)\mapsto f(\varphi(z))$ on the space of functions $\varphi:[0,1)\to I$. In the latter case, not only are the long-term behaviours of solutions critically dependent on the limit behaviour of the sequence $f^0(z), f(z), f^2(z),\dots$ (as in the discrete case) but also on the internal structure of $f^n(z),\ z\in I,$ as $n\to\infty$. Assuming f to be continuous, we consider the iterations of f as the semigroup $\langle f\rangle$ generated by f on the space of continuous maps, and introduce the notion of a limit semigroup for $\langle f\rangle$ in a wider map space in order to investigate asymptotic properties of $\langle f\rangle$. We construct a limit semigroup in the space of upper semicontinuous maps. This enables us to describe both of the aforementioned aspects of our interest around the iterations of f.

Keywords: One-Dimensional Map; Continuous-Time Difference Equation; Chaotic Dynamics

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1. Introduction

The study of continuous-time difference equations pave the way for the discovery of new kinds of complexity. These equations appear as models of systems whose future depends not only on the current state but also on part of the past history. Even the simplest nonlinear continuous-time difference equation

$$x(t)=f\left(x(t)\right),\quad t\in\mathbb{R}^+,\quad f \text{ is a continuous interval map,} \tag{1}$$

gives a wonderful example of how very complex behaviors can be described with the help of very simple models (for example, see [13, 14] and references therein). Over last decades, the focus of our research has been on this equation. As a result, the qualitative theory of equation (1) was completed and published not so long ago in [8] (the key ideas were put forward in [7, 11]).

Each solution of (1) is determined by its values on $[0,1)$ and can be written as

$$x(t) = f^n(\varphi(t-n)),\ t \in [n, n+1),\ n = 0, 1, \ldots,$$

$$\varphi(t) = x(t),\ t \in [0,1).$$

where $f^n(z) = f(f^{n-1}(z)),\ f^0(z) = z$. Typical solutions of (1), even solutions of any smoothness index, behave in bizarre ways, for example as in Figure 1.

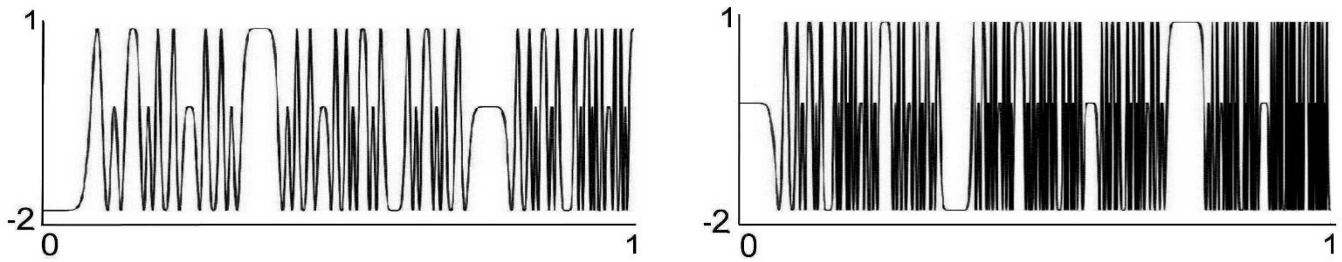

Figure 1. Typical solution of the equation $x(t+1) = f(x(t))$ with $x(t) = \varphi(t)$ for $t \in [0,1)$, where $f(z) = z^2 + \alpha,\ \varphi(t) = \alpha t,\ \alpha = -1{,}755$.

To describe the long-term properties of the solutions and explain how and why they occur, it is necessary to examine both the asymptotic properties of the iterative sequence $\{f^n(z)\}_n$ and the internal structure of the maps $f^n(z)$ by themselves for large n. As a rule, the structure of $f^n(z)$ becomes more and more intricate with the increase of n. The complexities in the behaviour of trajectories of the one-dimensional dynamical system $z \mapsto f(z)$ are transformed into a very complicated (even chaotic) structure of $f^n(z)$ as $n \to \infty$. But, at the same time, the sequence $\{f^n(z)\}_n$ is in a certain way (explained below) asymptotically periodic for almost all f.

With the aim to present in a uniform manner the findings in this direction, we considered the maps $f^0(z), f(z), f^2(z), \ldots, f^n(z) \ldots$ as the semigroup $\langle f \rangle$ generated by the map $f: I \to I$ in the space $C(I,I)$ of continuous maps of I into itself (with I being a closed bounded interval of the real line). If $\langle f \rangle$ is finite, then the sequence $\{f^n(z)\}_n$ is periodic and hence the structure of $f^n(z)$ is simple and repetitive as n increases. To study the general case where $\langle f \rangle$ is infinite, we developed the notion of a *limit semigroup* for $\langle f \rangle$ in a wider map space. We succeeded in constructing a limit semigroup in the space of upper semicontinuous maps (naturally, this space is the first thing that comes to mind because semicontinuous maps are pointwise limits of sequences of continuous maps). All this enabled us to achieve our aim as well as to discover some new insights.

The results just discussed was published only in Russian and, moreover, in their entirety only in [8]. Therefore, we thought it appropriate to cover them all in this article (with some unimportant simplifications and omissions for brevity).

2. Case of Finite Semigroup

In order to indicate conditions for $\langle f \rangle$ to be infinite, we introduce the set of maps

$$\Im = \{g \in C(I,I):\ g^2 = g\}.$$

Clearly, the set $\Im$ consists of continuous maps $g: I \to I$ such that for each one there exists an interval $[z_1, z_2] \subseteq I,\ z_1 \le z_2$, with the properties:

$$g(z) = z \text{ for } z \in [z_1, z_2],$$

$$g(z) \in [z_1, z_2] \text{ for } z < z_1 \text{ and } z > z_2.$$

If $g \in \Im$, then $g^n \in \Im$, $n = 2, 3, \ldots$. If in addition g is smooth, then either $g(z) = z$ or $g(z) \equiv \text{const}$.

Theorem 1 *The semigroup $\langle f \rangle$ is finite if and only if*

$$f^m \in \Im \quad \text{for a certain} \quad m \geq 1. \tag{2}$$

Proof. The `if' part is obvious since $f^{2m} = f^m$. Consider the `only if' part. If $\langle f \rangle$ is finite, then one can find $k, l \geq 1$ such that $f^{k+l} = f^k$. This gives $f^{ik+jl} = f^{ik}$ for all $i, j \geq 1$. Putting $i = l, j = k,$ we get $f^{2kl} = f^{kl}$, and consequently, $f^{kl} \in \Im$.

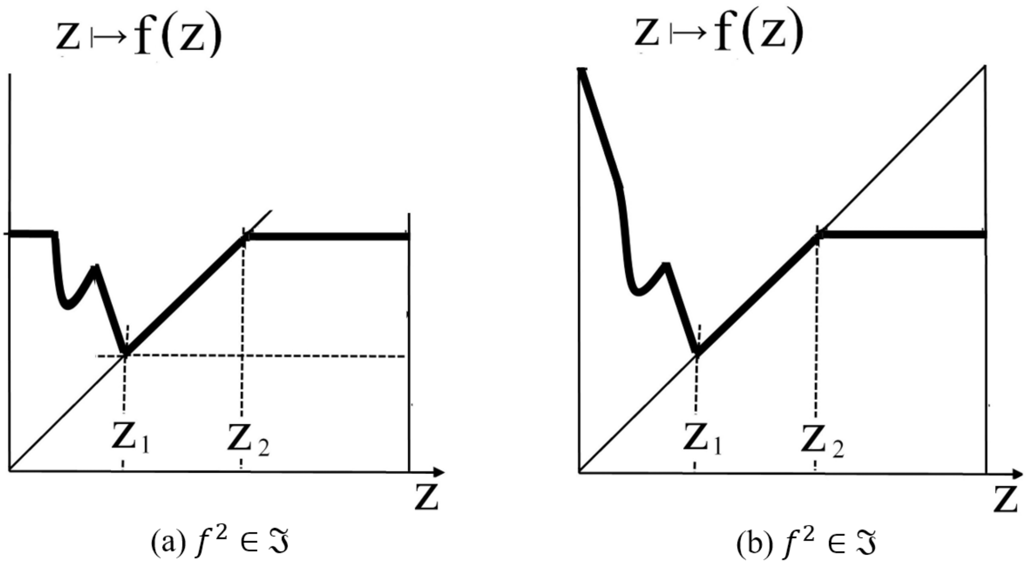

Figure 2. Maps that generate finite semigroups.

The dynamics of maps generating finite semigroups is very simple (see Figure 2). Let $\text{Fix}(f)$, $\text{Per}(f)$, and $\mathcal{D}(f)$ be the sets of fixed, periodic, and unstable points of f, respectively.

Theorem 2 *If $\langle f \rangle$ is finite, then $\mathcal{D}(f)$ is empty, $\text{Per}(f)$ is connected and hence* $\text{Per}(f) = \text{Fix}(f^2)$[1].

Proof. As f bing continuous, whatever $z \in I$, $\varepsilon > 0$, and $n > 0$, there exists $\delta_n = \delta_n(z, \varepsilon) > 0$ such that $|f^i(z) - f^i(z')| < \varepsilon$, $i = 1, \ldots, n$, if $|z - z'| < \delta_n$. As $\langle f \rangle$ is finite,

1 The converse theorem is not true, but the assertions ``$\mathcal{D}(f)$ is empty'' and ``$\text{Per}(f)$ is connected'' are equivalent.

$f^{2m} = f^m$ for a certain $m > 0$. Therefore $|f^i(z) - f^i(z')| < \varepsilon$ for all $i = 1, 2, \ldots$, if $|z - z'| < \delta_{2m}$, i.e., all points $z \in I$ are stable under f and $\mathcal{D}(f) = \varnothing$.

If $\mathcal{D}(f) = \varnothing$, then the set $\mathrm{Fix}(f)$ is connected: otherwise one can find $z', z'' \in \mathrm{Fix}(f)$ such that $(z', z'') \cap \mathrm{Fix}(f) = \varnothing$; therefore the function $f(z) - z$ maintains its sign in (z', z'') and hence one of the points z', z'' is unstable under f, which gives a contradiction. Since $\mathcal{D}(f) = \varnothing$ implies $\mathcal{D}(f^n) = \varnothing$, all the sets $\mathrm{Fix}(f^n)$, $n = 1, 2, \ldots$, are connected. Hence, if $z_1 < z_2 < \ldots < z_n$ form a cycle of f, then $[z_1, z_n] \subset \mathrm{Fix}(f^n)$, i.e. $f^n(z) = z$ for $z \in [z_1, z_n]$. As is known [1], this equality cannot hold for $n \geq 3$. Consequently, $\mathrm{Per}(f)$ coincides with $\mathrm{Fix}(f^2)$ and is therefore connected.

From Theorem 2 it follows that the set of continuous maps generating finite semigroups is exhausted by the maps f with the property: for f there exists its own interval $[z_1, z_2] \subseteq I$, $z_1 \leq z_2$, such that

$$\begin{aligned} &(a)\quad f(z) = z,\ z \in [z_1, z_2], \\ &\qquad f(z) > z_1,\ z < z_1 \quad \text{and} \quad f(z) < z_2,\ z > z_2, \\ &\qquad f^2(z) \neq z,\ z \notin [z_1, z_2], \\ &\text{or}\quad \text{(b) item (a) is met for } h = f^2; \end{aligned} \tag{3}$$

Two simplest maps that satisfy (3) are shown in Figure 2. In the case where the semigroup $\langle f \rangle$ is finite, its structure is easily derived from (3) by elementary calculations.

Theorem 3 *Let* $k = \min\{m > 0 : f^m \in \Im\}$. *Then* $\langle f \rangle$ *is finite semigroup that has:*

type $(k, 1)$, *if* $\mathrm{Fix}(f^2) = \mathrm{Fix}(f)$,

type $(k, 2)$, *if* $\mathrm{Fix}(f^2) \neq \mathrm{Fix}(f)$ *and* $f^{k-1}(I) \neq \mathrm{Fix}(f^2)$,

type $(k-1, 2)$, *if* $\mathrm{Fix}(f^2) \neq \mathrm{Fix}(f)$ *and* $f^{k-1}(I) = \mathrm{Fix}(f^2)$,

and $\langle f \rangle$ *is a group if and only if* f^2 *is the identity map* [2].

The map of Figure 2(a) generates a semigroup of type (1,1), and the map of Figure 2(b) generates a semigroup of type (1,2). Theorem 3 defines completely the finite semigroup $\langle f \rangle$. For example, if $\langle f \rangle$ has type $(k, 2)$, then $\langle f \rangle$ consists of the maps f^0, $f, \ldots,$ f^{k+1}, $f^{k+2} = f^k$, $f^{k+3} = f^{k+1}$, $f^{k+4} = f^k$, $\ldots$

3. Limit Semigroup Concept

The above section shows that the semigroup $\langle f \rangle$ is typically infinite. In this case, we will try to describe its asymptotic properties using another, simpler semigroup.

2 By the type of finite cyclic semigroup $\langle h \rangle$ is meant the pair (n, l), where n (index) and l (period) are the smallest positive integers such that $h^n = h^i$ for some positive integer $i \neq n$, and $h^k = h^{k+l}$.

Definition 1 Let the space $C(I,I)$ be embedded in a wider space $H_\nu(I,I)$ of maps $h:I\to I$, endowed with the metric ν. If there exists a periodic or almost periodic semigroup $\langle h\rangle$, $h\in H_\nu$, such that

$$\nu(f^n,\, h^n)\to 0 \quad \text{as} \quad n\to\infty, \tag{4}$$

then we say that $\langle h\rangle$ is the *limit semigroup* for $\langle f\rangle$ in H_ν.

Definition 2 The semigroup $\langle h\rangle$ is called *periodic* if there exists a positive integer N such that $h^{n+N}=h^n$, $n=0,1,\ldots,$ with N referring to as the period of $\langle h\rangle$ [3];

The semigroup $\langle h\rangle$ is called *almost periodic* if there are positive integers $N_1, N_2,\ldots$ such that for any $\varepsilon>0$ one can find N_k satisfying $\nu(h^{n+jN_k},h^n)<\varepsilon,\ n, j=0,1,\ldots,$ with $N_1,N_2,N_3,\ldots$ referring to as the almost-periods of $\langle h\rangle$.

Given H_ν, the semigroup $\langle f\rangle$ have just one limit semigroup or none at all. Taking for H_ν this or that map space and using the corresponding limit semigroup (if exists), one can describe the asymptotic properties of $\langle f\rangle$ in less or more detail.

In this connection, the question arises whether $\langle f\rangle$ can have a limit semigroup in ``its own'' space $C(I,I)$. From the above section it follows that every finite $\langle f\rangle$ has a limit semigroup in $C(I,I)$ and this is its largest subgroup. Indeed, let $\langle f\rangle$ be finite. Then the map

$$h_\star = f\circ f^\star, \quad \text{where} \quad f^\star = f^k,\ k=\min\{m>0: f^m\in\Im\}, \tag{5}$$

generates the semigroup $\langle h_\star\rangle,\ h\in C(I,I)$, which consists of the one element $f^\star$ if $\mathrm{Per}(f)=\mathrm{Fix}(f)$, and of the two elements $f^\star$ and $f\circ f^\star = f^{k+1}$ otherwise. Hence $\langle h_\star\rangle$ is a periodic group with period 1 or 2; moreover, $\langle h_\star\rangle$ is the largest subgroup of $\langle f\rangle$. By virtue of Theorem 3, $\| f^n - h_\star^n\|\to 0$ as $n\to\infty$. Consequently: *If $\langle f\rangle$ is finite, then its largest subgroup $\langle f\circ f^\star\rangle$ is the limit semigroup for $\langle f\rangle$ in $C(I,I)$.*

There is a representation for $f^\star$, in which k does not appear explicitly. Let $\omega_f(z)$ be for the ω-limit set of the trajectory of the point z under f. The map $f^\star$ can be written as

$$f^\star(z)=\begin{cases}\omega_f(z)=\lim\limits_{i\to\infty} f^i(z), & \text{if} \quad \mathrm{Per}(f)=\mathrm{Fix}(f),\\[2ex] \omega_{f^2}(z)=\lim\limits_{i\to\infty} f^{2i}(z), & \text{if} \quad \mathrm{Per}(f)\neq\mathrm{Fix}(f),\end{cases} \tag{6}$$

$$\text{furthermore} \quad f^\star\circ f^\star = f^\star \quad \text{and} \quad f\circ f^\star = f^\star\circ f. \tag{7}$$

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3 Thus, by a periodic semigroup we mean a semigroup of type $(1,N)$. Actually, $\langle h\rangle$ is a finite group that consists of the elements $h^0=h^N,\ h^1,\ldots h^{N-1}$ with h^0 being the unit element and h^{N-n} being the inverse element to h^n.

The map $f^\star$ in the form (6) can also exist for certain continuous maps that generate infinite semigroup. These are maps f with the property: for f there is its own interval $[z_1, z_2] \subseteq I$, $z_1 \le z_2$, such that

$$\begin{aligned} &(a')\ \ f(z) = z,\ z \in [z_1, z_2], \\ &\quad f(z) > z,\ z < z_1,\ \text{and}\ f(z) < z,\ z > z_2, \\ &\quad f^2(z) \ne z,\ z \notin [z_1, z_2], \\ &\text{or}\ \ (\text{b})\ \ \text{item}\ \ (\text{a})\ \text{is met for}\ h = f^2; \end{aligned} \tag{8}$$

An illustration is in Figure 3: although $\langle f_1 \rangle$ is finite and $\langle f_2 \rangle$ is infinite, (6) yields $f_1^\star = f_2^\star$.

If (8) is not met, then $\langle f \rangle$ no longer has a limit semigroup in $C(I, I)$ and we need a wider map space. We choose the space $SC(I, 2^I)$ of upper semicontinuous maps $h: I \to 2^I$, endowed with the metric

$$\rho_H(h_1, h_2) = \text{dist}_H(\text{gr}h_1, \text{gr}h_2),$$

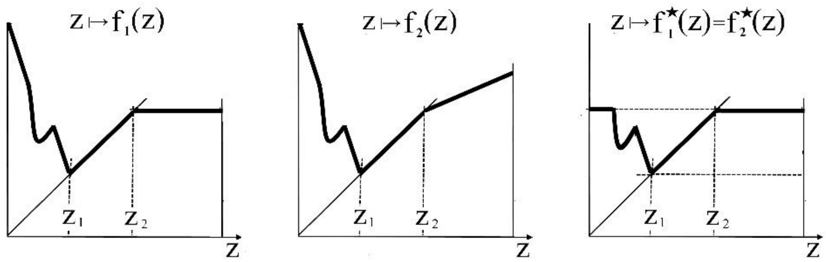


Figure 3. Maps that generate limit semigroups in the continuous map space.

where $\text{gr}h$ is for the graph of h and $\text{dist}_H(\cdot,\cdot)$ is for the Hausdorff distance between two sets [4]. The space $SC(I, 2^I)$ is compact in the topology induced by ρ_H, and $C(I, I)$ is embedded in $SC(I, 2^I)$.

Let Lim denote the operation of passing to the limit in $SC(I, 2^I)$. The meaning of the convergence in $SC(I, 2^I)$ becomes clear from the equivalence of the relations

$$\text{Lim}_{i\to\infty} h_i = h_* \quad \text{and} \quad \text{Lt}_{i\to\infty} \text{gr}h_i = \text{gr}h_*, \tag{9}$$

4 There are several equivalent definitions of upper semicontinuous maps, but for our purposes the most useful relates to the notion of map graph; namely, a map $h: I \to 2^I$, with 2^I being the set of all closed subsets of I, is said to be upper semicontinuous if its graph is a closed set in $I \times I$. The Hausdorff distance measures how far two subsets of a metric space are from each other and is given by $\text{dist}_H(A, B) = \max\{\sup_{a\in A} \rho(a, B), \sup_{b\in B} \rho(b, A)\}$, where A and B are closed bounded non-empty sets, which in our case are sets of the space $I \times I$ with Euclidean metric ρ.

where Lt is for the operation of passing to the topological limit. Thus, the convergence of maps in $SC(I,2^I)$ reduced to the existence of the topological limit of their graphs, which are treated as sets in the plane. It is therefore important to understand how the convergence of map graphs is related to the local behaviour of these maps. The answer is given by the following lemma.

Lemma 1 Let $\{g_i(w)\}_i$ be a sequence of maps from $C(J',J'')$ with J',J'' being closed bounded intervals, and let $V_\varepsilon(w)$ stand for the ε-neighborhood of $w \in J'$. The limit $\mathrm{Lt}_{i\to\infty}\,\mathrm{gr}\,g_i$ exists if and only if the limit $\mathrm{Lt}_{i\to\infty}\,g_i(V_\varepsilon(w))$ exists for any $w \in J'$ and any $\varepsilon > 0$ less than a certain $\varepsilon_w > 0$. Moreover, if so, then

$$\mathrm{Lt}_{i\to\infty}\,\mathrm{gr}\,g_i = G := \left\{(w,z) : w \in J', z \in G_w\right\}, G_w = \bigcap_{0<\varepsilon<\varepsilon_w} \mathrm{Lt}_{i\to\infty}\,g_i(V_\varepsilon(w)). \tag{10}$$

Proof. The 'if' part is proved by verifying that $\mathrm{Ls}_{i\to\infty}\,\mathrm{gr}\,g_i \subseteq G \subseteq \mathrm{Li}_{i\to\infty}\,\mathrm{gr}\,g_i$ with Ls and Li being for the upper and lower topological limits, respectively. Take, say, the latter inclusion. If $(w,z) \in G$, then $z \in \mathrm{Lt}_{i\to\infty}\,g_i(V_\varepsilon(w))$, $\varepsilon < \varepsilon_w$, i.e., for any $\varepsilon < \varepsilon_w$ there is a sequence of points $z_{i,\varepsilon} \in J''$ such that $z_{i,\varepsilon} \in g_i(V_\varepsilon(w))$ and $z_{i,\varepsilon} \to z$ as $i \to \infty$. Let us consider the sequence $\{z_{i,1/i}\}_i$, $i > 1/\varepsilon_w$. Since $z_{i,1/i} \in \varphi_i(V_{1/i}(w))$, one can find $w_i \in V_{1/i}(w)$ such that $g(w_i) = z_{i,1/i}$ and $w_i \to w$ as $i \to \infty$. Therefore $(w_i, g(w_i)) \to (w,z)$ as $i \to \infty$, and hence $(w,z) \in \mathrm{Li}_{i\to\infty}\,\mathrm{gr}\,g_i$ as claimed.

The 'only if' part follows from the definition of topological limit. If $\mathrm{Lt}_{i\to\infty}\,\mathrm{gr}\,g_i$ exists, then $\mathrm{Lt}_{i\to\infty}\,g_i(V_\varepsilon(w))$ also exists for every given $w \in J'$ and sufficiently small $\varepsilon > 0$. Hence, the set G_w is well defined and, by what was just shown, (10) holds.

The composition $h' \circ h''$ of $h', h'' \in SC(I,2^I)$ is understood as the map $h' \circ h'' = \bigcup_{u \in h''(z)} h'(u)$, which obviously belongs to $SC(I,2^I)$. Let $h \in C(I,I)$ and $h_i \in SC(I,2^I)$, $i = 1,2,\ldots$ From the topological sense of the convergence in $SC(I,2^I)$, it follows that, if $\mathrm{Lim}_{i\to\infty} h_i$ exists, then $\mathrm{Lim}_{i\to\infty} h \circ h_i$, and $\mathrm{Lim}_{i\to\infty} h_i \circ h$ also exist, moreover,

$$\mathrm{Lim}_{i\to\infty} h \circ h_i = h \circ h_*, \quad \text{where} \quad h_* = \mathrm{Lim}_{i\to\infty} h_i. \tag{11}$$

The equality $\mathrm{Lim}_{i\to\infty} h_i \circ h = h_* \circ h$ is usually not satisfied.

4. Resolvent Map

Now the goal is to construct a special map $f^\Delta \in SC(I,2^I)$ so as $f \circ f^\Delta$ to generates a limit semigroup for $\langle f \rangle$ (of course, f^Δ must be identical to $f^\star$, if $\langle f \rangle$ is finite). We call f^Δ the *resolvent map* for $\langle f \rangle$ and accept that

$$f^\Delta = \mathrm{Lim}_{k\to\infty} f^{k!}. \tag{12}$$

So far, this is only a formal notation for f^Δ, since the existence of the limit in (12) is not obvious. If f^Δ exists, then it belongs to the space $SC(I,2^I)$, where 2^I is the subset of 2^I, consisting of closed intervals (including degenerated ones).

Let $\mathrm{Per}(f) = \mathrm{Fix}(f)$. Then $\mathrm{Lt}_{i\to\infty} f^i(V)$ exists for any connected $V \subset \hat{I}$ (see [6]) and, by Lemma 1, the sequences $\{f^i\}_i$ and $\{f^{k!}\}_k$ converge in $SC(\hat{I},\hat{I})$ to the same limit. Therefore $f^\Delta = \mathrm{Lim}_{i\to\infty} f^i$; an example is in Figure 4. If $\mathrm{Per}(f) = \mathrm{Fix}(f^s)$, $s > 0$, similar arguments lead to the formula $f^\Delta = \mathrm{Lim}_{i\to\infty} f^{is}$ and $f^\Delta = f^\star$ for finite $\langle f \rangle$.

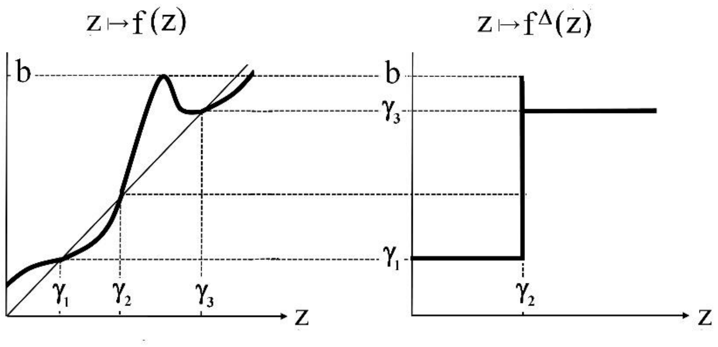
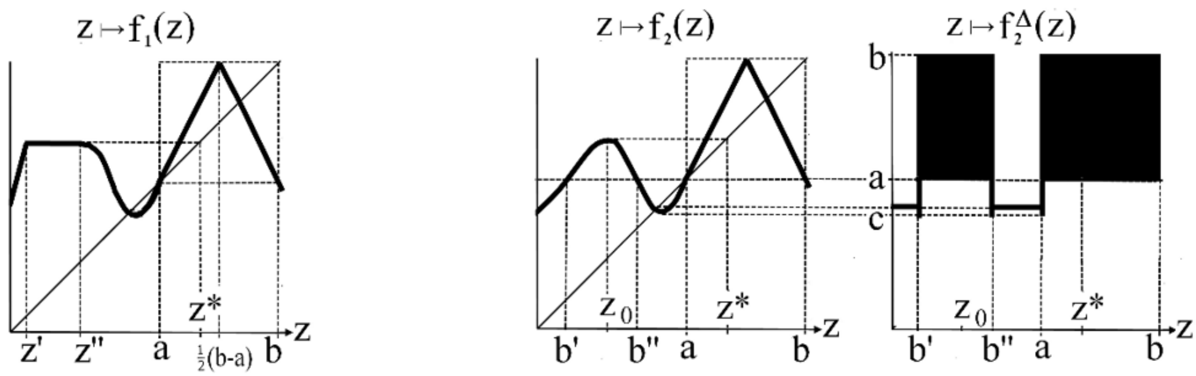


Figure 4. Resolvent map f^Δ in the case where $\mathrm{Per}(f) = \mathrm{Fix}(f)$.

Where $\mathrm{Per}(f) \neq \mathrm{Fix}(f^s)$ for any $s > 0$, the resolvent map f^Δ may not exist. The simplest example occurs when some interval $[z', z''] \subset I$ collapses to a point, say, z^* under f. Then $\mathrm{Lt}_{k\to\infty}\, \mathrm{gr}\, f^{k!}|_{[z',z'']} = \lim_{k\to\infty} f^{k!}(z^*)$ when the limit on the right-hand side exists. If the trajectory of z^* is neither asymptotically periodic nor asymptotically almost-periodic, then the sequence $\{f^{k!}\}_k$ diverges and f^Δ cannot exist. An example is delivered by the maps drawn in Figure 5. For each of f_1 and f_2, almost all points of $[a,b]$ are not asymptotically periodic or asymptotically almost-periodic. Thereby, f_1^Δ is non-existent for almost all z* (because $[z', z'']$ collapses to z^* under f_1) and f_2^Δ exists for all z^* regardless of whether z has the periodicity property or not, namely, $f_2^\Delta(z) = [a,b]$. Trajectories of almost all points of $[a,b]$ are dense in $[a,b]$ under both f_1 and f_2. This is what ensures the existence of f_2^Δ, but for f_1^Δ this argument does not ``work'' whenever z^* does not have periodicity properties under f_1.

Thus, in order for f^Δ to exists, f must be subject to further restrictions. Before stating the theorem on the resolvent map, we need few more notions.



(a) Resolvent map does not exist.

(b) Resolvent map exists.

Figure 5. Examples on the question of the existence of resolvent maps.

Definition 3 The set

$$Q_f(z) = \bigcap_{\delta>0} \bigcap_{j>0} \overline{\bigcup_{i>j} f^i\left(V_\delta(z)\right)},\ z \in I,\ V_\delta(z) = (z-\delta, z+\delta) \cap I, \tag{13}$$

is called the *domain of influence* [5] of the point z under the map f.

The set $Q_f(z)$ is nothing but the upper topological limit of the sets $f^i(V_\delta(z))$, $i = 0,1,\ldots$, as $i \to \infty$ and $\delta \to 0$. Consequently, $Q_f(z)$ shows how far from the trajectory of the point z the trajectories of its nearby points go. For example, in Figure 4, the fixed point $z = \gamma_1$ is attracting and the fixed point $z = \gamma_2$ is repelling. Therefore, $Q_f(\gamma_1)$ is the one-point set $\{\gamma_1\}$ (moreover, $Q_f(z) = \{\gamma_1\}$ for all z that are attracted to $z = \gamma_1$), and $Q_f(\gamma_2)$ is the interval $[\gamma_1, b]$.

Let $\omega_f(z)$ stand for the ω-limit set of the trajectory of z under f. Clearly, $Q_f(z)$ contains $\omega_f(z)$. More precisely: if $z \notin \mathrm{D}(f)$, then $Q_f(z) = \omega_f(z)$ and $Q_f(z)$ has no interior points, but if $z \in \mathrm{D}(f)$, then at least one of its half-neighborhoods ``expands'' under f, therefore the interior of $Q_f(z)$ is nonempty and $Q_f(z) \supseteq \omega_f(z)$.

The notion of the domain of influence of a point produces the notion of the domain of influence of a set:

$$Q_f(A) = \bigcap_{\delta>0} \bigcap_{j>0} \overline{\bigcup_{i>j} f^i\left(V_\delta(A)\right)},\ V_\delta(A) \text{ is the } \delta\text{-neighborhood of } A \subset I,$$

which, of course, is equivalent to the formula $Q_f(A) = \bigcup_{z \in \overline{A}} Q_f(z),\ A \subset I$.

Definition 4 A non-degenerated interval is called a ξ*-interval* of f, if the trajectories of all its points have the same ω-limit set that is different from a cycle or the closure of an almost periodic trajectory.

The simplest variant of ξ-intervals is an interval that collapses (under f) into a point whose trajectory is neither asymptotically periodic nor asymptotically almost periodic, as in Figure 5(b).

Definition 5 An ordered set of intervals $J_0, J_1, \ldots, j_{N-1}$ is said to be a *cycle of intervals* of the map f with period N if these intervals are cyclically permutated by f and have mutually no common interior points.

Having introduced these definitions, we can pass to the resolvent map.

Theorem 4 *If the map f has no ξ-intervals, then the resolvent map f^Δ exists and it can be written in the form*

$$f^\Delta(z) = \bigcap_{k>0} Q_{f^{k!}}(z),\ z \in I. \tag{14}$$

Proof. The full proof is too long to include here, but the idea is that the theorem will be confirmed (in view of Lemma 1) if we will show that for any $z \in I$ there is $\varepsilon_z > 0$ such that $\mathrm{Lt}_{k\to\infty} f^{k!}(V_\varepsilon(z))$ exists for $0 < \varepsilon < \varepsilon_z$ and

5 This notion was introduced in [11] as a development of the well-known notion of prolongation. Somewhat later, it was employed by other authors, but without using the very term ``domain of influence''.

$$\bigcap_{0<\varepsilon<\varepsilon_z} \mathrm{Lt}_{k\to\infty} f^{k!}(V_\varepsilon(z)) = \bigcap_{k>0} Q_{f^{k!}}(z). \tag{15}$$

We consider only the simplest case where $\omega_f(z)$ is an attracting cycle. If so, then: $Q_{f^i}(z) = \omega_{f^i}(z)$ for $i = 1, 2, \dots$; $\mathrm{Lt}_{j\to\infty} f^{jN}(z) = \omega_{f^N}(z)$ with N being the period of the attracting cycle, and there exists $\varepsilon_z > 0$ such that $\omega_{f^N}(z') = \omega_{f^N}(z)$ for $z' \in V_\varepsilon(z)$ if $\varepsilon < \varepsilon_z$. Consequently,

$$\mathrm{Lt}_{j\to\infty} f^{jN}(V_\varepsilon(z)) = \mathrm{Lt}_{j\to\infty} f^{jN}(z) = \omega_{f^N}(z) = Q_{f^N}(z) \text{ if } \varepsilon < \varepsilon_z.$$

As $\{f^{k!}(z)\}_{k\geq N}$ is a subsequence of $\{f^{jN}(z)\}_j$, we obtain (15) from the equalities

$$\mathrm{Lt}_{k\to\infty} f^{k!}(V_\varepsilon(z)) = Q_{f^N}(z) \text{ if } \varepsilon < \varepsilon_z, \text{ and } Q_{f^{k!}}(z) = Q_{f^N}(z).$$

In case $\omega_f(z)$ is the closure of an almost-periodic trajectory, the proof is similar but more technically involved, it is presented in [11]. A proof that covers all cases can be found in [8], so far only in Russian.

Theorem 4 will hold if

(i) *$\omega_f(z)$ is a cycle or the closure of an almost-periodic trajectory for all $z \in I$*. It is known [10, 12] that (i) certainly holds if $\mathrm{Per}(f)$ is closed (then $\omega_f(z)$ is a cycle whatever $z \in I$) and fails if f has a cycle with a non-power-of-two period. Furthermore, (i) is no longer guaranteed if f has cycles of all periods $1, 2, 2^2, \dots$ but no others [6]

Here is another sufficient condition for Theorem 4

(ii) $Q_f(z)$ is a cycle or a cycle of intervals for all $z \in I$.

If (ii) is modified as follows

(iii) *$Q_f(z)$ is a cycle or a cycle of intervals for all $z \in I$ and their periods are uniformly bounded,*

then f^Δ takes the very simple form

$$f^\Delta = \mathrm{Lim}_{k\to\infty} f^{2pk} \quad \text{and} \quad f^\Delta(z) = Q_{f^{2p}}(z), \tag{16}$$

where $p < \infty$ is the least common multiple of the periods of all $Q_f(z)$, $z \in I$.

So, the resolvent map f^Δ may or may not exist, and when it does, it is usually arranged complexly. The main properties of f^Δ are listed below.

6 Among such maps there are maps (even C^∞-smooth ones) that have a non-closed set of almost periodic point, and for them (i) does not hold (besides, the resolvent map does not exist). Moreover, the assumption of the closedness of the set of almost periodic points still does not provide (i) (see [12]). But it is highly probable that in the class of C^1-smooth maps this assumption would be sufficient for (i) to hold.

- The map $f^{\Delta}(z)$, thought of as a function form I into 2^{I}, is upper semicontinuous. Its value is a non-degenerated closed interval for $z \in \mathrm{D}(f)$ and a one-point set (singleton) for $z \in I \setminus \mathrm{D}(f)$.
- The map $f^{\Delta}(z)$, thought of as a function form I into I, is single-valued and continuous over $I \setminus \mathrm{D}(f)$, and is multi-valued over $\mathrm{D}(f)$.

Definition 6 The set of values that $f^{\Delta}(z)$ takes on for $z \in \mathrm{D}(f)$ is called its *spectrum of jumps* and denoted by $\mathrm{J}(f^{\Delta})$.

Two simple illustrations are given by the maps of Figure 4 and Figure 5(b): here $\mathrm{J}(f^{\Delta})$ consists of the interval $[\gamma_1, b]$ and $\mathrm{J}(f_2^{\Delta})$ consists of the intervals $[a,b]$ and $[c,b]$.

- The spectrum of jumps $\mathrm{J}(f^{\Delta})$ is composed from the connected component of domains of influence of the unstable points of f. Any two elements of $\mathrm{J}(f^{\Delta})$ either do not have common interior points, or one of them contains the other.
- In typical situations, $\mathrm{J}(f^{\Delta})$ as well as the set of values of $f^{\Delta}(z)$ on $I \setminus \mathrm{D}(f)$ are finite (in particular, this is the case for structurally stable f).
- In many cases, the graph of f^{Δ} is locally self-similar and, furthermore, fractal: $\mathrm{gr}\, f^{\Delta}$ is a plane curve whose fractal dimension is greater than 1.

The first four properties are easy to obtain from Theorem 4, and the last one needs some comments. Let $\dim_{\mathrm{box}}$ denote the box-counting dimension (one of the commonly used versions of fractal dimension). It is clear that

$$\dim_{\mathrm{box}} \mathrm{gr}\, f^{\Delta} = \dim_{\mathrm{box}} \mathrm{D}(f) + 1$$

usually holds (at least when $\mathrm{J}(f^{\Delta})$ is finite). The set $\mathrm{D}(f)$ is known to be often fractal. Therefore, if $\dim_{\mathrm{box}} \mathrm{D}(f) \neq 0$, then the fractal dimension of $\mathrm{gr}\, f^{\Delta}$ is greater than 1 and, moreover, it can even be equal to 2 (the latter is always the case where $\mathrm{D}(f)$ contains an interval).

More details and arguments about f^{Δ} can be found in [8], but here the properties of f^{Δ} are illustrated in Figure 6, which displays the resolvent map for the logistics parabola

$$\eta : z \to \lambda z(1-z), \quad z \in [0,1], \quad \lambda > 0 \text{ - parameter.} \tag{17}$$

The dynamical properties of (17) are well known, so we use them without explanations.

If $\lambda = 3.5$, then η has an attracting period-4 cycle, which attracts all points of [0,1] except for the set $\mathrm{D}(\eta)$, which consists of the repelling period-1 and period-2 periodic points of η and their inverse images, and is therefore countable and nowhere dense. The resolvent map $\eta^{\Delta}(z)$, thought of as a function $[0,1] \to [0,1]$, is piecewise constant over $[0,1] \setminus \mathrm{D}(\eta)$ and takes on four values that correspond to the points of the attracting cycle, but over $\mathrm{D}(\eta)$, it becomes interval-valued with the spectrum of jumps consisting of three intervals (see Figure 6(a)).

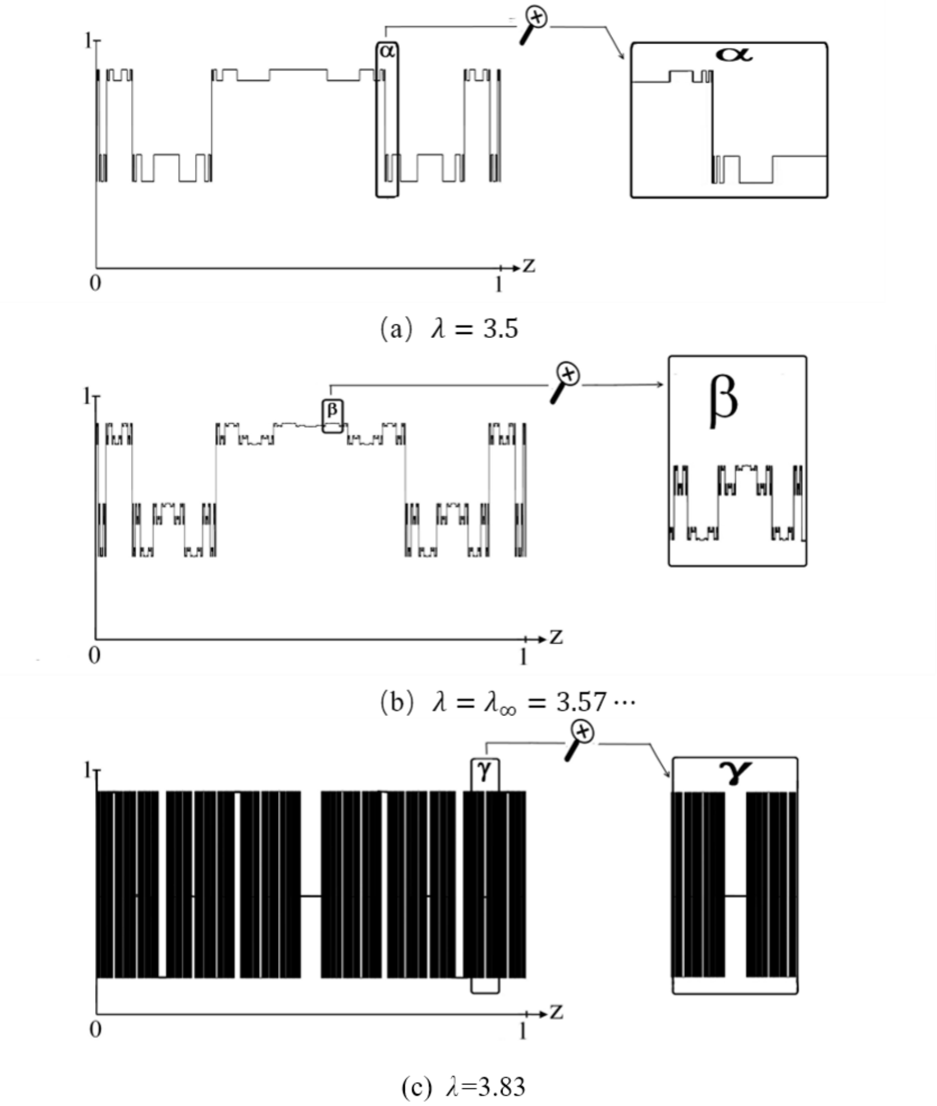


Figure 6. Resolvent maps for $\eta : z \mapsto \lambda z(1-z)$.

Where $\lambda = 3.83$ and η has an attractive period-3 cycle and repelling cycles of any period, the properties of $\eta^{\Delta}(z)$ are nearly the same as in the previous case (see Figure 6(c)). The only difference is that $D(\eta)$, over which $\eta^{\Delta}(z)$ is interval-valued, is already uncountable but again nowhere dense (now $D(\eta)$ is a Cantor-like set, since it is the closure of the repelling periodic points of η and their inverse images). In addition, $D(\eta)$ has positive fractal dimension and, hence, the graph of $\eta^{\Delta}(z)$ is a fractal.

Now a very different situation where $\lambda = \lambda_{\infty}(\approx 3.57)$ with λ_{∞} being the parameter value at which η has only cycles with periods $1, 2, 2^2, \ldots$ (see Figure 6(b)). These cycles are repelling and their inverse images are dense on $[0,1]$. Hence, $D(\eta)$ is a countable dense set and all points of $[0,1] \setminus D(\eta)$ are attracted to the Cantor-like set $\mathcal{K}$ consisting of the condensation points of $\mathrm{Per}(\eta)$. It follows that $\eta^{\Delta}(z)$ is singleton-valued over $[0,1] \setminus D(\eta)$ and its values run through the continual set $\mathcal{K}$; whereas $\eta^{\Delta}(z)$ is interval-valued over

$D(\eta)$ and its spectrum of jumps is countable (the infimum of the lengths of intervals from $J(\eta^{\Delta})$ equals 0).

Graphics shown in Figure 6 have local self-similarity in the vicinity of each ``largest'' vertical segment in case (a), and each vertical segment in cases (b) and (c). Moreover, in the last case, there are an uncountable number of vertical segments, which leads to the fractality of the graph.

5. Limit Semigroup in the Space of Upper Semicontinuous Maps

The existence of f^{Δ} given by (12) does not in itself guarantee that $f \circ f^{\Delta}$ generates the limit semigroup for $\langle f \rangle$. The last is automatically true if

$$f^{\Delta} \circ f^{\Delta} = f^{\Delta} \text{ and } f^{\Delta} \circ f = f \circ f^{\Delta} \tag{18}$$

(the former of these equalities entails the latter, but not the other way around).

The condition of Theorem 4, generally speaking, do not imply (18). A confirming example comes from Figure 7. For f depicted there, f^{Δ} exists, but f and f^{Δ} do not commute: $f^{\Delta}(a) = [c,\gamma]$, $f(f^{\Delta}(a)) = [c,\gamma]$, $f^{\Delta}(f(a)) = f^{\Delta}(\gamma) = [c,\gamma_2]$, and hence $f^{\Delta} \circ f = f \circ f^{\Delta}$. Besides, $f^{\Delta} \circ f^{\Delta} \neq f^{\Delta}$. Here, either of the equalities in (18) fails because f has an extremum at the unstable point $z_{\circ}$.

There are maps for which only the former equality in (18) breaks down. One example are

$$g : z \to \begin{cases} z^2, & z \in [0,1), \\ (z+1)/2, & z \in [1,2], \end{cases} \text{ and } g^{\Delta} : z \to \begin{cases} 0, & z \in [0,1), \\ [0,1], & z = 1, \\ 1, & z \in (1,2] \end{cases}$$

(the drawing is left to the reader). Here, g has the semi-attracting fixed point $z=1$, which leads to that the domain of influence of a point $z \in [1,2]$ does not coincide with the domain of influence of its ω-limit set, namely: $Q_g(z) = \omega_g(z) = \{1\}$ and $Q_g(\omega_g(z)) = [0,1]$. Therefore, $g^{\Delta}(z) = 1$ and $g^{\Delta}(g^{\Delta}(z)) = [0,1]$ for $z \in [1,2]$. In order for the last equality in (18) to be also violated, we ``improve'' g by replacing it with

$$g : z \to \begin{cases} z^2, & z \in [0,1), \\ 1, & z \in [1,2], \end{cases}$$

which has much the same dynamical property as g. More precisely: $Q_{\bar{g}}(z) = \omega_{\bar{g}}(z) = \{1\}$ and $Q_{\bar{g}}(\omega_{\bar{g}}(z)) = [0,1]$ for $z \in (1,2]$. As a result, $\bar{g}^{\Delta} = g^{\Delta}$, but $\bar{g}^{\Delta} \circ \bar{g} \neq \bar{g} \circ \bar{g}^{\Delta}$, whereas $g^{\Delta} \circ g = g \circ g^{\Delta}$. Indeed, $\bar{g}^{\Delta}(\bar{g}(z)) = [0,1]$ and $\bar{g}(\bar{g}^{\Delta}(z)) = \{1\}$ for $z \in [1,2]$.

There are smooth maps for which (18) is not satisfied, for example, $z \mapsto 4z^2(1-z)$.

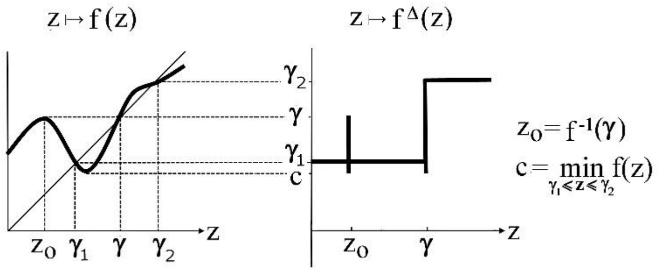


Figure 7. Example of a map that does not commutate with its associated resolvent map.

As the above examples demonstrate, violating (18) occurs when the set

$$\mathcal{Q}_{\neq}(f) = \{z \in I : Q_{f^s}(z) \neq Q_{f^s}(\omega_{f^s}(z)) \text{ for some integer } s = s(z) > 0\}$$

is non-empty. It is not difficult to realize that this condition is necessary and sufficient.

Theorem 5 *Let f satisfy the condition of Theorem 4. The equalities (18) take place if and only if $\mathcal{Q}_{\neq}(f) = \varnothing$.*

Proof. The 'only if' part is evident. The 'if' part follows from $f^n(Q_{f^m}(z)) = Q_{f^m}(f^n(z))$ and $Q_{f^n}(Q_{f^m}(z)) = Q_{f^{r(n,m)}}(z)$ with $r = (n,m)$ being the greatest common divisor of n and m, because these formulas are valid only provided that $\mathcal{Q}_{\neq}(f) = \varnothing$.

The condition $\mathcal{Q}_{\neq}(f) = \varnothing$ does not seem very good for practical use, but it is much easier to check it where f satisfies (iii) and, hence, f^{Δ} is given by (16) with some $p < \infty$. It suffices to analyse just f^{2p} and not all the iterations of f .[7]

Based on Theorem (5), from now on we will assume that

(v) the map f has no ξ-intervals,
(vv) $\mathcal{Q}_{\neq}(f) = \varnothing$,

and refer to (v),(vv) as the *limit semigroup conditions* or LSG-conditions for short.

These conditions cover most of continuous intervals maps, in particular, structurally stable maps. For instance, the logistic parabola (17) is structurally stable (and, hence, satisfies LSG-conditions) on a parameter set containing an open subset; this subset consists of those λ for which there is an attracting cycle with multiplier less than 1 in absolute value. Of course, LSG-conditions can be met by maps that are not structurally stable. An example is again given by the logistic parabola. If $\lambda = 3$, then η has the attracting fixed point $z = 2/3$ whose multiplier is equal to -1. In this case, $p = 1$ and

7 This follows from the relations: if $s > 2p$, then $Q_{f^s}(z) = Q_{f^{s(\mathrm{mod}\,2p)}}(z)$; but if $s < 2p$, then $Q_{f^s}(z) = Q_{f^{2p}}(z)$ when s is not a divisor of $2p$, and $Q_{f^s}(z) = \bigcup_{j=0}^{\frac{2p}{s}-1} f^{js}(Q_{f^{2p}}(z))$ otherwise.

$Q_{\eta^2}(z) = Q_{\eta^2}(\omega_{\eta^2}(z)) = \begin{cases} \{2/3\}, & z \in (0,1), \\ [0,3/4], & z = 0,\ z = 1, \end{cases}$ (see Figure 8), and thus the LSG-conditions are met.

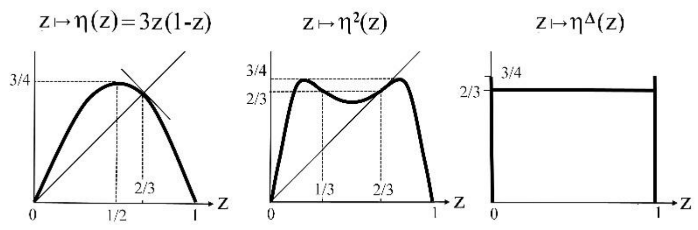


Figure 8. Example of a structurally unstable map that satisfies LSG-conditions.

Once LSG-conditions are valid, f^{Δ} exists and $f \circ f^{\Delta}$ generates in $SC(I, 2^I)$ the semigroup $\langle f \circ f^{\Delta} \rangle$ that consists of just the maps~ $h^n = f^n \circ f^{\Delta}$, $n = 0,1,2,\ldots$ Then from (11) and (18) it follows that

$$f^n \circ f^{\Delta} = f^{\Delta} \circ f^n \ \text{ and } \ f^n \circ f^{\Delta} = \mathrm{Lim}_{k\to\infty} f^{n+k!}, \quad n = 0,1,\ldots, \tag{19}$$

with the limit being uniform in n.

Theorem 6 *Let LSG-conditions be met.*

1. *The semigroup $\langle f \circ f^{\Delta} \rangle$ is periodic or almost periodic.*
2. *The semigroup $\langle f \circ f^{\Delta} \rangle$ is the limit semigroup for $\langle f \rangle$ in the space $SC(I, 2^I)$.*

Proof. We consider the simple case where f satisfies (iii). Then f^{Δ} is written in the form (16) with some $p < \infty$. Therefore $f^{2p} \circ f^{\Delta} = f^{\Delta}$ and $f^{n+2p} \circ f^{\Delta} = f^n \circ f^{\Delta}$, $n = 0,1,\ldots$. Hence $\langle f \circ f^{\Delta} \rangle$ is periodic, as claimed in item 1. Item 2 can be considered true if we succeed in proving that

$$\rho_H(f^n, f^n \circ f^{\Delta}) \to 0 \quad \text{as} \quad n \to \infty. \tag{20}$$

It is clear that $f^n \circ f^{\Delta} = f^s \circ f^{\Delta}$ with some $0 \le s \le 2p-1$, $s = s(n)$. Since $f^s \circ f^{\Delta} = \mathrm{Lim}_{k\to\infty} f^{s+2pk}$, for any $\varepsilon > 0$ there exists $K(\varepsilon, s) > 0$ such that $\rho_H(f^{s+2pk}, f^s \circ f^{\Delta}) \le \varepsilon$ if $k > K(\varepsilon, s)$. Hence,

$$\rho_H(f^n, f^n \circ f^{\Delta}) \le \varepsilon \ \text{ for } n > 2p(K+1), \ K = \max_{0\le s\le 2p-1} K(\varepsilon, s),$$

which gives (20), and item 2 follows. If (iii) is not satisfied, the proof becomes more complicated and needs some additional background. It can be found in [8].

If $\langle f \rangle$ is finite (in which case f satisfies (iii) with $p = 1$ or $p = 2$), then its limit semigroup $\langle f \circ f^{\Delta} \rangle$ is periodic with period 1 or 2. At the same time, $\langle f \rangle$ is typically not periodic. It is periodic only if f^2 is the identity map, i.e., if either $f(z) = z$, or $f(z) = -z + 2c_\circ$ with $c_\circ$ being the midpoint of I. Indeed, in both the cases, $f^{-1}(z) = f(z)$ and therefore $\langle f \rangle$ is actually periodic with period 1 (in the former case) or 2 (in the latter one); moreover, $f^{\Delta} = f^{\star}$ and $f^{\star}$ is the identity map, i.e., $\langle f \circ f^{\Delta} \rangle$ coincides with $\langle f \rangle$, which means that $\langle f \rangle$ is the limit semigroup for itself in $C(I,I)$ as well as in $SC(I,2^I)$.

There is another approach to the study of transformation groups, which based on the notion of Ellis enveloping semigroup [2-4]. For $\langle f \rangle$, the enveloping semigroup $E(f)$ is defined as the closure of the set $\{f^n : I \to I,\ n = 1,2,\ldots\}$ in the space I^I of maps of I into I, equipped with the pointwise convergence topology. The asymptotic properties of $\langle f \rangle$ are described by the following subset of $E(f)$:

$$E'(f) = \left\{ h \in I^I : \exists n_i \to \infty \setminus \mathrm{h}(z) = \lim_{i\to\infty} f^{ni}(z), z \in I \right\}.$$

Where $E'(f)$ is a finite semigroup, it can be regarded in a certain sense as an analogue of the limit semigroup for $\langle f \rangle$ in $C(I,I)$. Indeed, $E'(f)$ is finite if and only if

$$\mathrm{Per}(f) = \mathrm{Fix}(f^{2^m}) \ \text{ for some integer } m > 0. \tag{21}$$

Then $E'(f)$ is a periodic group of period 2^m, namely,

$$\mathrm{E}'(f) = \langle f \circ f_{\star} \rangle, \ \text{ where } f_{\star}(z) = \lim_{i\to\infty} f^{i\cdot 2^m}(z). \tag{22}$$

This statement is a direct consequence of the following well-known fact: the trajectories of all $z \in I$ are asymptotically periodic under f with their asymptotic periods being uniformly bounded, if and only if (21) holds (see [9]). In this case, it is easy to verify that $f_{\star}(z) = f^{\star}(z)$ and $f^{\Delta}(z) = f^{\star}(z)$ for $z \notin \mathrm{D}(f)$. Hence, $f_{\star}$ is identical to the resolvent map over $I \setminus \mathrm{D}(f)$. In addition, $f_{\star}$ is continuous just if $\mathrm{D}(f) = \varnothing$ (or, equivalently, $\mathrm{Per}(f)$ is connected) and then $f_{\star} = f^{\star}$ over I. Otherwise, $f_{\star}(z)$ is a first Baire class function. Thus, the limit semigroup approach, compared with the enveloping semigroup approach, can give more information about the asymptotic properties of $\langle f \rangle$ and allows one to study a wider class of maps f.

6. Limit Sequence

LSG-conditions (v),(vv) are always satisfied by structurally stable maps, and then the limit semigroup $\langle f \circ f^{\Delta} \rangle$ exists and consists of the maps

$$f^{\Delta},\ f \circ f^{\Delta},\ f^2 \circ f^{\Delta},\ \ldots. \tag{23}$$

When f is not structurally stable, LSG-conditions may not be met. If this is due to the failure of only (vv), then f^{Δ} exists, but its associated maps (23) can no longer form a semigroup. Nevertheless, it may happen that the maps (23) constitute a periodic or almost periodic sequence and still retain the principal property of limit semigroups

$$\rho_H(f^n, f^n \circ f^{\Delta}) \to 0 \ \text{ as } n \to \infty. \tag{24}$$

If so, the maps (23) continue to describe the asymptotic properties of $\langle f \rangle$. In particular, such a situation occurs where f satisfies (iii). Then (v) holds and the sequence (23) is periodic. Consequently, (24) holds true regardless of whether (vv) holds or not.

As an example, we again take the logistic parabola $\eta: z \mapsto \lambda_* z(1-z)$, where the parameter value $\lambda_* (\approx 3.678)$ is chosen so that the critical point $z = c = 1/2$ is mapped onto the repelling fixed point $z = \gamma$ in three iterations, i.e., $\eta^3(c) = \gamma$ (see Figure 9(b)). In this case, the domain of influence of each $z \in [0,1]$ is an interval. Namely, setting $c_1 = \eta(c)$ and $c_2 = \eta^2(c)$, we get that $Q_\eta(0) = Q_\eta(1) = [0, c_1]$ and $Q_\eta(z) = [c_2, c_1]$ for $z \neq 0,1$, where $[c_2, c_1]$ absorbs all $z \in (0,1)$, each after finitely many iterations, and $[c_2, c_1]$ can be split into two adjoint intervals: $J_1 = [\gamma, c_1]$ and $J_2 = [c_2, \gamma]$ that passe into each other under η (see Figure 9(a)). It follows that η, which is structurally unstable, satisfies (iii), and hence η^Δ exists (see Figure 9(b)) and has the form (16) with $p = 1$, namely,

$$\eta^\Delta(z) = Q_{\eta^2}(z) = \begin{cases} [0, c_1], & z = 0,\ z = 1; \\ J_n, & z \in \cup_{j \geq 0} \operatorname{int} f^{-2j}(J_n),\ n = 1,2; \\ J_1 \cup J_2, & \text{in other cases.} \end{cases}$$

But the condition (vv): $\mathcal{Q}_{\neq}(\eta) = \varnothing$ is violated because of the critical point $z = c = 1/2$. Indeed, $Q_{\eta^2}(c) = J_1$ and $Q_{\eta^2}(\omega_{\eta^2}(c)) = Q_{\eta^2}(\gamma) = J_1 \cup J_2$. Therefore the critical point belongs to $\mathcal{Q}_{\neq}(\eta)$ (the same goes for its inverse images). The result is that

$$\eta(\eta^\Delta(c)) = J_2, \quad \eta^\Delta(\eta(c)) = \eta^\Delta(c_1) = J_1 \cup J_2, \quad \text{and hence} \quad \eta^\Delta \circ \eta \neq \eta \circ \eta^\Delta.$$

But even so, the ``limit'' property (24) holds for η (this can be checked directly or by using $\eta^n(Q_{\eta^n}(z)) = Q_{\eta^n}(z)$). Since the sequence $\{\eta^n \circ \eta^\Delta\}_n$ is periodic with period 2.

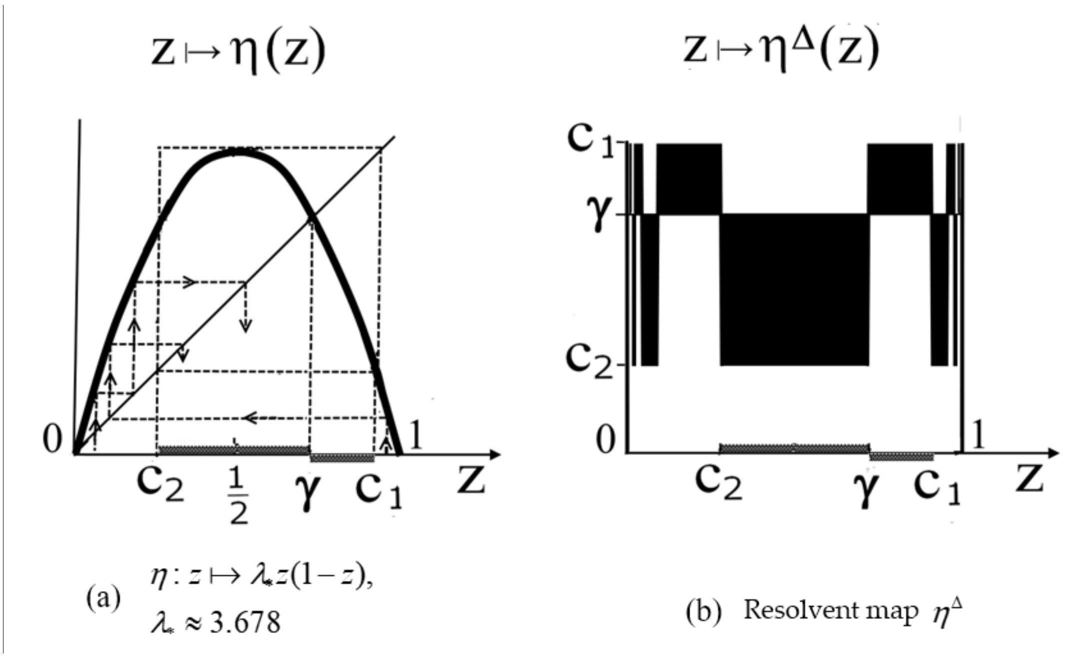


Figure 9. Example of a map that generates a semigroup does not having a limit semigroup (although its associated resolvent map exists).

Based on these facts, we arrive at the following generalization of the limit semigroup notion. To the iterative sequence $id, f, f^2, \ldots$, whose elements form the semigroup~$\langle f \rangle$, we assign the sequence (23) and call it the *limit sequence* in $SC(I, I)$ if it is periodic or almost periodic and has the ``limit'' property (24).

As the proof of Theorem 6 shows, the sequence (23) has the property (24) only if $\mathrm{Lim}_{k\to\infty} f^{n+k!}$ is uniform in $n=1,2,\ldots$ The uniformity of the limit is ensured by LSG-conditions (v),(vv). But they are not necessary, in particular, if f satisfies (iii), then the uniformity follows from the periodicity of (23) (here (v) is met automatically and (vv) is optional). Thus, generally speaking, (24) needs verification.

Interestingly, the ``alternative'' sequence $f^{\Delta}, f^{\Delta}\circ f, f^{\Delta}\circ f^{2}, \ldots$ cannot be regarded as a limit sequence, of course except where $f^{\Delta}\circ f = f\circ f^{\Delta}$.

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