

Article

The Black-Scholes Exotic Barrier Option Pricing Formula

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Abstract: The paper considers a specific type of such financial instrument as an option, namely an exotic barrier call option of the European type. Exotic options are gaining popularity among ordinary investors due to the development of information and telecommunication technologies, thanks to which such specific financial instruments as options have become readily available. We investigate the hedging problem for such options with some restrictions on the payment function and the availability of dividend payment on a risky asset in the classical Black-Scholes model. An analogue of the Black-Scholes formula for the mentioned variant of the exotic barrier is proved. In the future, it is planned to generalize the obtained results for put options and for more general payment functions.

Keywords: Black-Scholes Model, Financial Market, Exotic Barrier Option, Payment Function, Pricing, Hedging

1. Introduction

P. Samuelson in [1] proposed the geometric Brownian motion as a model for dynamics of a financial asset price. The Samuelson model is widely known and studied and it became one of the most important models of financial mathematics. On its basis, F. Black and M. Scholes in [2] calculated the "fair" option values. R. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model and coined the term "Black-Scholes options pricing model" [3].

The main principle behind the model is to hedge the option by buying and selling the underlying asset in a specific way to eliminate risk. This type of hedging is called "continuously revised delta hedging" and is the basis of more complicated hedging strategies such as those engaged in by investment banks and hedge funds.

The Black-Scholes options pricing model is widely used, although often with some adjustments, by options market participants. The Black-Scholes formula has only one parameter that cannot be directly observed in the market: the average future volatility of the underlying asset, though it can be found from the price of other options. Since the option value (whether put or call) is increasing in this parameter, it can be inverted to produce a "volatility surface" that is then used to calibrate other models, e.g. for OTC derivatives.

R. Merton and M. Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work, the committee citing their discovery of the risk-neutral dynamic revision as a breakthrough that separates the option from the risk of the underlying security. Although ineligible for the prize because of his death in 1995, F. Black was mentioned as a contributor by the Swedish Academy.

Many results have been obtained by researchers for different types of options and various conditions on payment function and other parameters of the Black-Scholes options pricing model. In contrast to existing results, we consider this model for an exotic

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barrier European-type call option with some restrictions on the payment function and we prove an analogue of the Black-Scholes formula for such options.

2. Exotic options

We investigate the Black-Scholes options pricing model for special kinds of options – so-called "exotic barrier options". Let us briefly describe exotic options using [4].

Exotic options are a category of options contracts that differ from traditional options in their payment structures, expiration dates, and strike prices. The underlying asset or security can vary with exotic options allowing for more investment alternatives. Exotic options are hybrid securities that are often customizable to the needs of the investor.

Exotic options are a variation of the American and European style options—the most common options contracts available. American options let the holder exercise their rights at any time before or on the expiration date. European options have less flexibility, only allowing the holder to exercise on the expiration date of the contracts. Exotic options are hybrids of American and European options and will often fall somewhere in between these other two styles.

A traditional options contract gives a holder a choice or right to buy or sell the underlying asset at an established price before or on the expiration date. These contracts do not obligate the holder to transact the trade.

The investor has the right to buy the underlying security with a call option, while a put option provides them the ability to sell the underlying security. The process where an option converts to shares is called exercising, and the price at which it converts is the strike price.

An exotic option can vary in terms of how the payoff is determined and when the option can be exercised. These options are generally more complex than plain vanilla call and put options.

Exotic options usually trade in the over-the-counter (OTC) market. The OTC marketplace is a dealer-broker network, as opposed to a large exchange such as the New York Stock Exchange (NYSE).

Further, the underlying asset for an exotic can differ greatly from that of a regular option. Exotic options can be used in trading commodities such as lumber, corn, oil, and natural gas as well as equities, bonds, and foreign exchange. Speculative investors can even bet on the weather or price direction of an asset using a binary option.

Barriers in the exotic options are determined by the underlying price and ability of the stock to be active or inactive during the trade period, for instance, up-and-out option has a high chance of being inactive should the underlying price go beyond the marked barrier.

Down-and-in-option is very likely to be active should the underlying prices of the stock go below the marked barrier. Up-and-in option is very likely to be active should the underlying price go beyond the marked barrier. One-touch double barrier binary options are path-dependent options in which the existence and payment of the options depend on the movement of the underlying price through their option life.

There are a lot of papers devoted to exotic options. A more detailed review of different results for exotic options one can find in [5-8].

3. Assumption of the Black-Scholes model

The Black-Scholes (or Black-Scholes-Merton) model is a mathematical model for the dynamics of a financial market containing derivative investment instruments. This model assumes that the market consists of at least one riskless asset, usually called the money market, cash, or *bond*, and one risky asset usually called *the stock*.

The following assumptions are made about the assets (which relate to the names of the assets):

- Riskless rate: The rate of return on the riskless asset is constant and thus called the risk-free interest rate.
- Random walk: The instantaneous log return of the stock price is an infinitesimal random walk with drift; more precisely, the stock price follows a geometric Brownian motion, and it is assumed that the drift and volatility of the motion are constant. If drift and volatility are time-varying, a suitably modified Black–Scholes formula can be deduced, as long as the volatility is not random.
- The stock does not pay a dividend.
- The assumptions about the market are:
 - No arbitrage opportunity (i.e., there is no way to make a riskless profit).
 - Ability to borrow and lend any amount, even fractional, of cash at the riskless rate.
 - Ability to buy and sell any amount, even fractional, of the stock (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).

The described model is called the Bond & Stock market or financial (B, S) –market. With these assumptions, suppose there is a derivative security also trading in this market. It is specified that this security will have a certain payoff at a specified date in the future, depending on the values taken by the stock up to that date. Even though the path the stock price will take in the future is unknown, the derivative's price can be determined at the current time. For the special case of a European call or put option, F. Black and M. Scholes showed that "it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock" [2]. Their dynamic hedging strategy led to a partial differential equation that governs the price of the option. Its solution is given by the Black–Scholes formula.

4. The Black–Scholes formula

On the complete probability space (Ω, F, F_t, P) let's consider financial (B, S) –market. In this market, we consider a risk-free asset (bank deposit, government bonds, etc.) whose current price is described by the ordinary differential equation

$$dB_t = rB_t dt, \quad (1)$$

where $r > 0$ is the annualized risk-free interest rate.

The solution of equation (1) with initial condition $B_0 > 0$ is a function

$$B_t = B_0 e^{-rt}.$$

A risky asset (a stock) is also considered, the evolution of the price of the base of which is described by the stochastic Ito differential equation:

$$dS_t = S_t(\mu dt + \sigma dw_t). \quad (2)$$

Here, $\mu \geq 0$ is the drift coefficient, $\sigma > 0$ is the diffusion coefficient.

It follows from Ito's formula that the solution of equation (2) with initial condition $S_0 > 0$ is a stochastic process

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w_t\right\}, \quad (3)$$

where w_t is the standard Wiener process that is adapted to the filtration $F_t = \sigma(w_s, s \leq t)$.

We will assume that all assumptions of the model (described in section 3 above) are fulfilled, in particular that (B, S) – market is arbitrage-free.

Then the density of the martingale measure P_t^* with respect to the initial measure P_t has the form

$$Z_t = \frac{dP_t^*}{dP_t} = \exp \left\{ -\frac{\mu - r}{\sigma} w_t - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 t \right\}.$$

It follows from the Girsanov theorem that the process

$$w_t^* = w_t + \left(\frac{\mu - r}{\sigma} \right) t \quad (4)$$

is Wiener process with respect to the martingale measure P^* .

Let's rewrite the value of the basic asset (3) in terms of the new Wiener process w_t^* . To do this, we substitute expression (4) in formula (3). As a result, we get

$$\begin{aligned} S_t &= S_0 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma w_t \right\} = S_0 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \left(w_t^* - \left(\frac{\mu - r}{\sigma} \right) t \right) \right\} \\ &= S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma w_t^* \right\}. \end{aligned}$$

Here and further we use such notations:

- r is the annualized risk-free interest rate, continuously compounded (also known as the force of interest).
- S_t is the price of the underlying asset at time t .
- $C(S_t, t)$ is the price of a European call option.
- T is the time of option expiration.
- τ is the time until maturity: $\tau = T - t$
- K is the strike price of the option, also known as the exercise price.
- $F(x)$ denotes the standard normal cumulative distribution function:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp \left\{ -\frac{z^2}{2} \right\} dz.$$

The value of a European call option for a non-dividend-paying underlying stock in terms of the Black–Scholes parameters is (the Black–Scholes formula for a European call option):

$$C(S_t, t) = S_t F(d_1) - K e^{-r(T-t)} F(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma \sqrt{T-t}} \left[\ln \left(\frac{S_t}{K} \right) + \left(r + \frac{\sigma^2}{2} \right) (T-t) \right], \\ d_2 &= \frac{1}{\sigma \sqrt{T-t}} \left[\ln \left(\frac{S_t}{K} \right) + \left(r - \frac{\sigma^2}{2} \right) (T-t) \right]. \end{aligned}$$

5. Main result for exotic barrier options

Barrier options are exotic options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.

A number of different types of barrier options regularly trade in the over-the-counter market. They are attractive to some market participants because they are less expensive than the corresponding regular options. These barrier options can be classified as either knock-out options or knock-in options. A knock-out option ceases to exist when the

underlying asset price reaches a certain barrier; a knock-in option comes into existence only when the underlying asset price reaches a barrier.

A down-and-out call is one type of knock-out option. It is a regular call option that ceases to exist if the asset price reaches a certain barrier level H . The barrier level is below the initial asset price. The corresponding knock-in option is a down-and-in call.

This is a regular call that comes into existence only if the asset price reaches the barrier level.

On the specified financial (B, S) – market we consider exotic barrier call option with barrier level H , option parameters K_1, K_2, K_3, K_4 and with a payment function such as below

$$f(S_t) = \max\{(K_1 - S_t), K_2\} I\{S_t < H\} + \max\{(K_3 - S_t), K_4\} I\{S_t \geq H\}, \quad (5)$$

where $I\{A\}$ is the indicator function of the set A , i.e.

$$I\{S_t < H\} = \begin{cases} 1, & \text{if } S_t < H; \\ 0, & \text{otherwise} \end{cases}$$

and

$$K_1 - K_2 \leq H \leq K_3 - K_4; \quad K_1 \geq K_2; \quad K_3 \geq K_4.$$

Let's consider financial (B, S) – market and let's assume that all assumptions of the Black–Scholes–Merton model (described in section 3 above) are fulfilled.

The main result of the paper is the following theorem.

Theorem 1. Let the exotic barrier option be given by the payment function of the form (5). Then its value is determined by the formula:

$$C(S_t, t) = (K_1 - K_2)A_{K_1 - K_2}^- - S_t \left(F(A_{K_1 - K_2}^+) + F(A_{K_3 - K_4}^+) + F(A_H^+) \right) + \exp\{-r\tau\} \\ \left[K_2 F(A_H^-) - F(A_{K_1 - K_2}^-)(K_1 - K_2) + K_3 \left(F(A_{K_3 - K_4}^-) - K_2 F(A_H^-) \right) + K_4 \left(1 - F(A_{K_3 - K_4}^-) \right) \right], \quad (6)$$

where we denote

$$A_f^\pm = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{f}\right) - \left(r \pm \frac{\sigma^2}{2} \right) \tau \right)$$

Proof of Theorem 1. According to the general theory of financial obligations [5, 6, 8], the value of an option with a payment function of the form (5) has the form

$$C(S_t, t) = C_1(\tau, S_t, K_1, K_2, K_3, K_4, H, r, \sigma) = E^* e^{-r\tau} f(S_t) = \\ = E^* e^{-r\tau} \left(\max\{(K_1 - S_t), K_2\} I\{S_t < H\} + \max\{(K_3 - S_t), K_4\} I\{S_t \geq H\} \right) = \\ = E^* e^{-r\tau} \max\{(K_1 - S_t), K_2\} I\{S_t < H\} + E^* e^{-r\tau} \max\{(K_3 - S_t), K_4\} I\{S_t \geq H\} = \\ = D_1(\tau, S_t, K_1, K_2, H, r, \sigma) + D_2(\tau, S_t, K_3, K_4, H, r, \sigma) = D_1 + D_2,$$

where we denote (here E^* is the mathematical expectation with respect to a measure P^*)

$$D_1 = E^* e^{-r\tau} \max\{(K_1 - S_t), K_2\} I\{S_t < H\},$$

$$D_2 = E^* e^{-r\tau} \max\{(K_3 - S_t), K_4\} I\{S_t \geq H\}.$$

Given that

$$\max\{(K_1 - S_t), K_2\} I\{S_t < H\} = \begin{cases} K_1 - S_t, & \text{if } S_t < K_1 - K_2; \\ K_2, & \text{if } K_1 - K_2 \leq S_t < H; \\ 0, & \text{if } S_t \geq H; \end{cases}$$

let's obtain the mathematical expectation in expression for D_1 .

We get

$$D_1 = \frac{\exp\{-r\tau\}}{\sqrt{2\pi}} \int_{-\infty}^{A_1} \left(K_1 - S_t \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}x\right\} \right) \exp\left\{-\frac{x^2}{2}\right\} dx + \\ + K_2 \frac{\exp\{-r\tau\}}{\sqrt{2\pi}} \int_{A_1}^{A_2} \exp\left\{-\frac{x^2}{2}\right\} dx,$$

where we denote

$$A_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{K_1 - K_2}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau \right), \\ A_2 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{H}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau \right).$$

Further let us rewrite D_1 in the following form

$$D_1 = \exp\{-r\tau\} \left[K_1 F(A_1) - \frac{S_t}{\sqrt{2\pi}} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\tau\right\} \int_{-\infty}^{A_1} \exp\left\{-\frac{x^2 - 2\sigma\sqrt{\tau}x}{2}\right\} dx + \right. \\ \left. + K_2 (F(A_2) - F(A_1)) \right] = \exp\{-r\tau\} K_2 F(A_2) + \exp\{-r\tau\} F(A_1) (K_1 - K_2) - \\ - \exp\{-r\tau\} \frac{S_t}{\sqrt{2\pi}} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2\tau}{2}\right\} \int_{-\infty}^{A_1} \exp\left\{-\frac{(x - \sigma\sqrt{\tau})^2}{2}\right\} dx = \\ = \exp\{-r\tau\} K_2 F(A_2) + \exp\{-r\tau\} F(A_1) (K_1 - K_2) - \\ - \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{A_1} \exp\left\{-\frac{(x - \sigma\sqrt{\tau})^2}{2}\right\} dx = \\ = \exp\{-r\tau\} K_2 F(A_2) + \exp\{-r\tau\} F(A_1) (K_1 - K_2) - \\ - S_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1 - \sigma\sqrt{\tau}} \exp\left\{-\frac{y^2}{2}\right\} dy = \\ = \exp\{-r\tau\} K_2 F(A_2) + \exp\{-r\tau\} F(A_1) (K_1 - K_2) - S_t F(A_1 - \sigma\sqrt{\tau}).$$

Now let's denote

$$A_3 = A_1 - \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{K_1 - K_2}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau \right) - \sigma\sqrt{\tau} = \\ = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{K_1 - K_2}\right) - \left(r + \frac{\sigma^2}{2}\right)\tau \right).$$

So we get

$$D_1 = \exp\{-r\tau\} K_2 F(A_2) + \exp\{-r\tau\} F(A_1)(K_1 - K_2) - S_t F(A_3).$$

Next, similarly

$$\max\{(K_3 - S_t), K_4\} I\{S_t \geq H\} = \begin{cases} 0, & \text{if } S_t < H; \\ K_3 - S_t, & \text{if } H \leq S_t < K_3 - K_4; \\ K_4, & \text{if } S_t \geq K_3 - K_4. \end{cases}$$

From which it follows that D_2 can be represented in the form

$$D_2 = \frac{\exp\{-r\tau\}}{\sqrt{2\pi}} \int_{A_2}^{A_4} \left(K_3 - S_t \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\tau + 2\sigma\sqrt{\tau}x\right\} \right) \exp\left\{-\frac{x^2}{2}\right\} dx + \\ + \frac{K_4 \exp\{-r\tau\}}{\sqrt{2\pi}} \int_{A_3}^{+\infty} \exp\left\{-\frac{x^2}{2}\right\} dx,$$

where we denote

$$A_4 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{K_3 - K_4}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau \right).$$

Continuing the equality above, we get

$$D_2 = \exp\{-r\tau\} K_3 (F(A_4) - F(A_2)) + \exp\{-r\tau\} K_4 (1 - F(A_4)) - \\ - \frac{\exp\{-r\tau\}}{\sqrt{2\pi}} \int_{A_2}^{A_4} \left(K_3 - S_t \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\tau + 2\sigma\sqrt{\tau}x\right\} \right) \exp\left\{-\frac{x^2}{2}\right\} dx = \\ = \exp\{-r\tau\} K_3 (F(A_4) - F(A_2)) + \exp\{-r\tau\} K_4 (1 - F(A_4)) - \\ - S_t (F(A_5) + F(A_6)),$$

where we denote

$$A_5 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{K_3 - K_4}\right) - \left(r + \frac{\sigma^2}{2}\right)\tau \right), \\ A_6 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{S_t}{H}\right) - \left(r + \frac{\sigma^2}{2}\right)\tau \right).$$

Thus, we can conclude that the value of the barrier exotic option with payment function given by the (5) can be determined by the formula given in the statement of the theorem:

$$C(S_t, t) = (K_1 - K_2) A_{\bar{K}_1 - K_2} - S_t \left(F(A_{K_1 - K_2}^+) + F(A_{K_3 - K_4}^+) + F(A_H^+) \right) + \\ + \exp\{-r\tau\} \left[K_2 F(A_H^-) - F(A_{\bar{K}_1 - K_2}^-) (K_1 - K_2) + K_3 \left(F(A_{\bar{K}_3 - K_4}^-) - K_2 F(A_H^-) \right) \right. \\ \left. + K_4 \left(1 - F(A_{\bar{K}_3 - K_4}^-) \right) \right].$$

Theorem 1 is proved.

6. Conclusions and prospects for further research

Basically, the Black-Scholes model is used in the following cases:

- to find undervalued options for sale and overvalued options for purchase;
- for portfolio hedging, which allows to reduce risks in case of low volatility;
- to assess market conditions for future volatility values.

As a rule, the Black-Scholes model is used by traders to compare current and theoretical values of option prices. If the theoretical value does not coincide with the current one and the difference between them is greater than the value of the transaction, then traders use arbitrage tactics on this difference. However, the model is based on a theory that predicts the absence of arbitrage. In this regard, in fact, the Black-Scholes model uses several people who find and crowd out situations in the market with arbitrage. It is worth noting that this assumption is considered fully justified.

Another way to use the model is based on calculating hedge positions for a portfolio of stocks. Due to the fact that the price fluctuations of the options coincide with the price of the share, the sale of options allows you to balance losses from the share. For this, the Black-Scholes model is used, which determines the number of put options to achieve the desired volatility.

In addition, the Black-Scholes model is used to calculate market conditions for volatility (σ). In this case, it is assumed that the market has correctly valued the options, so the formula can easily be used to find the market estimate of the lower and upper bounds of the share price in the future. Narrow distribution curves are constructed from these values, which increase the probability of the theoretical price approaching its future value. In other words, the higher the stake, the greater the difference in price expectations.

It is also worth noting that in the case of revaluation of the option, the Black-Scholes model is used to find quantitative probabilities that are determined by market expectations. Despite the fact that traders mainly use one model algorithm, different values can be substituted into the formula.

"Sigma" is calculated based on previous market data, which can be taken from any moment. As a rule, the calculation is based on data for the last year, so using a shorter or longer time interval leads to different results. As a result, it turns out that the Black-Scholes model cannot become a panacea for traders, it acts only as a very valuable tool that allows you to evaluate options and market expectations.

The proposed model of barrier options pricing will be enhanced using recent studies of applications of random processes [9-17] and computer technologies [15-19].

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