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Article

Quantum Properties of Coherently Driven Three-Level Atom Coupled to Vacuum Reservoir

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Abstract: A three-level laser with an open cavity and a two-mode vacuum reservoir is explored for its quantum properties. Our investigation begins with a normalized order of the noise operators associated with the vacuum reservoir. The master equation and linear operators' equations of motion are used to determine the equations of evolution of the atomic operators' expectation values. The equation of motion answers are then used to calculate the mean photon number, photon number variance, and quadrature variance for single–mode cavity light and two–mode cavity light. As a result, for γ=0, the quadrature variance of light mode **a** is greater than the mean photon number for two-mode cavity light. As a result, for the two-mode cavity light, the maximum quadrature squeezing is 43.42 percent.

Keywords: Atom; Photon Statistics; Vacuum Reservoir; Squeezing; Quadrature Variance

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1. Introduction

A quantum optical system in which light is created by three-level atoms inside a cavity coupled to a vacuum reservoir is known as a three-level laser. A source of coherent or chaotic light emitted by an atom inside a cavity coupled to a vacuum reservoir is known as a three-level atom [1]. A three-level atom's top, intermediate, and bottom levels are designated by $|a\rangle_\kappa$, $|b\rangle_\kappa$, and $|c\rangle_\kappa$, respectively. Due to stimulated or spontaneous emission, a three-level atom in the top level may decay to level $|b\rangle$ and eventually to level $|c\rangle$.

The purpose of this paper is to investigate the squeezing and statistical properties of light produced by a coherently driven three-level atom in an open cavity connected to a two-mode vacuum reservoir through a single-port mirror. We also determined equations of evolution of the atomic operators' expectation values using the master equation and large-time approximation. The mean photon number, photon number variance, and quadrature variances of single-mode cavity light beams were calculated using the derived solutions. We calculated the mean photon number, photon number variance, and quadrature squeezing of the two-mode cavity light using the same approaches. We perform our calculations by conventionally grouping the noise operators connected with the vacuum reservoir [1, 2].

2. Dynamics of Linear Operators

As shown in the Figure 1, the atoms' top, intermediate, and bottom levels are indicated by $|a\rangle_\kappa$, $|b\rangle_\kappa$, and $|c\rangle_\kappa$, respectively. When an atom transitions from level a to level b and from level b to level c, two photons with the same or different frequencies are emitted, with direct transitions between levels a and c being completely prohibited. When an atom transitions from the top to the intermediate level, light mode a is emitted, whereas light mode b is emitted when the atom transitions from the intermediate to the bottom level [3, 4].

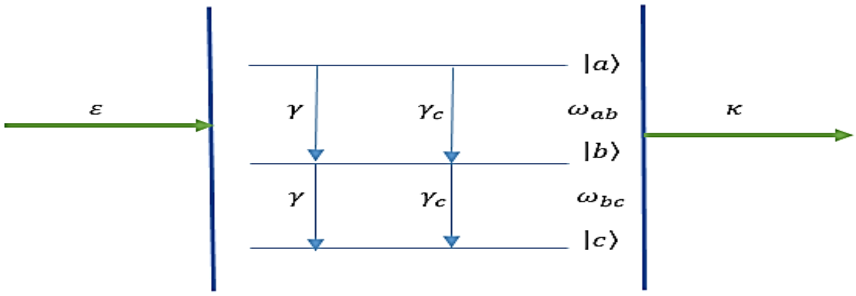

Figure 1. The model

The Hamiltonian describes how coherent light couples the top and bottom levels of a three-level atom [5, 6],

$$\hat{H}_{1=}\frac{i\Omega}{2}[\hat{\sigma}_c^\dagger - \hat{\sigma}_c], \quad (1)$$

where

$$\hat{\sigma}_c = |c\rangle\langle a|, \quad (2)$$

is a lowering operator and

$$\Omega = 2\varepsilon\lambda. \quad (3)$$

Here, λ is the coupling constant between the driving coherent light and a three-level atom, and ε is the amplitude of the driving coherent light, which is considered to be real and constant. The interaction of light modes a and b with the atom at resonance is represented by the Hamiltonian [2, 7, 8]

$$\hat{H}_2 = ig(\hat{\sigma}_a^\dagger\hat{\sigma} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b), \quad (4)$$

where

$$\hat{\sigma}_a = |b\rangle\langle a|, \quad (5)$$

and

$$\hat{\sigma}_b = |c\rangle\langle b|, \quad (6)$$

are lowering atomic operators, g is the coupling constant between the atom and cavity mode **a** or **b**, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b. Thus, the total Hamiltonian is given by

$$\hat{H} = \frac{i\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + ig(\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + +\hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b). \quad (7)$$

The master equation for a three-level atom interacting with a two-mode vacuum reservoir has the form [5]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a) + \frac{\gamma}{2}(2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b), \quad (8)$$

where γ is the spontaneous emission decay constant. Hence with the aid of equation (2) the master equation can be put in the form

$$\frac{d\hat{\rho}}{dt} = \frac{\gamma}{2}(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a) + \frac{\gamma}{2}(2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b)$$

$$+\frac{\Omega}{2}\left(\hat{\sigma}_c^{\dagger}\hat{\rho}-\hat{\sigma}_c\hat{\rho}-\hat{\rho}\hat{\sigma}_c^{\dagger}+\hat{\rho}\hat{\sigma}_c\right)+\frac{\gamma}{2}\left(2\hat{\sigma}_b^{\dagger}\hat{\rho}\hat{\sigma}_b^{\dagger}-\hat{\eta}_b\hat{\rho}-\hat{\rho}\hat{\eta}_b\right)+\frac{\gamma}{2}\left(2\hat{\sigma}_a^{\dagger}\hat{\rho}\hat{\sigma}_a^{\dagger}-\hat{\eta}_a\hat{\rho}-\hat{\rho}\hat{\eta}_a\right). \quad (9)$$

When the noise operators associated with the vacuum reservoir are set in normal order and the noise forces have no effect on the dynamics of the cavity mode operators, the equations of motion for the operators a and b assume the form [1, 8, 9]

$$\frac{d\hat{a}}{dt}=-\frac{\kappa}{2}\hat{a}-i[\hat{a},\hat{H}], \quad (10)$$

and

$$\frac{d\hat{b}}{dt}=-\frac{\kappa}{2}\hat{b}-i[\hat{b},\hat{H}]. \quad (11)$$

Here, κ, is the cavity damping constant and considered to be the same for cavity modes **a** and **b**. Then in view of equation (7), equations of motion for the operators $\hat{a}$ and $\hat{b}$ turn out to be

$$\frac{d\hat{a}}{dt}=-\frac{\kappa}{2}\hat{a}-g\hat{\sigma}_a, \quad (12)$$

and

$$\frac{d\hat{b}}{dt}=-\frac{\kappa}{2}\hat{b}-g\hat{\sigma}_b. \quad (13)$$

Upon adding equations (12) and (13), we get

$$\frac{d\hat{c}}{dt}=-\frac{\kappa}{2}\hat{c}-g(\hat{\sigma}_a+\hat{\sigma}_b), \quad (14)$$

where,

$$\hat{c}=\hat{a}+\hat{b}, \quad (15)$$

is the annihilation operator for the superposition of light modes **a** and **b**. By employing the relation [8, 10]

$$\frac{d}{dt}\langle\hat{A}\rangle=\mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{A}\right), \quad (16)$$

we btain

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle=g\mathrm{Tr}\left[\hat{\rho}\hat{\eta}_b\hat{a}-\hat{\rho}\hat{a}\hat{\eta}_a+\hat{\rho}\hat{b}^{\dagger}\hat{\sigma}_c\right]+\frac{\Omega}{2}\mathrm{Tr}\left(\hat{\rho}\hat{\sigma}_b^{\dagger}\right)+\frac{\gamma}{2}\mathrm{Tr}(-\hat{\rho}\hat{\sigma}_a)+\frac{\gamma}{2}\mathrm{Tr}(-\hat{\rho}\hat{\sigma}_a)$$

$$=g\left[\langle\hat{\eta}_a\hat{a}\rangle-\langle\hat{\eta}_a\hat{\eta}_a\rangle+\langle\hat{b}^{\dagger}\hat{\sigma}_c\rangle\right]+\frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger}\rangle-\gamma\langle\hat{\sigma}_a\rangle. \quad (17)$$

With the same procedure one can obtain the following

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle=g\left[\langle\hat{\eta}_c\hat{b}\rangle-\langle\hat{\eta}_b\hat{b}\rangle-\langle\hat{a}^{\dagger}\hat{\sigma}_c\rangle\right]-\frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger}\rangle-\frac{\gamma}{2}\langle\hat{\sigma}_b\rangle, \quad (18)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle=g\left[\langle\hat{\sigma}_b\hat{a}\rangle-\langle\hat{\sigma}_a\hat{b}\rangle\right]-\frac{\Omega}{2}\left[\langle\hat{\eta}_c\rangle-\langle\hat{\eta}_a\rangle\right]-\frac{\gamma}{2}\langle\hat{\sigma}_c\rangle, \quad (19)$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = g\left[\langle\hat{\sigma}_a^{\dagger}\hat{a}\rangle + \langle\hat{a}_a^{\dagger}\hat{\sigma}_a\rangle\right] + \frac{\Omega}{2}\left[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^{\dagger}\rangle\right] - \gamma\langle\hat{\eta}_a\rangle, \tag{20}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = g\left[\langle\hat{\sigma}_b^{\dagger}\hat{a}\rangle + \langle\hat{b}^{\dagger}\hat{\sigma}_b\rangle - \langle\hat{\sigma}_a^{\dagger}\hat{a}\rangle - \langle\hat{a}^{\dagger}\hat{\sigma}_a\rangle\right] + \gamma[\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle], \tag{21}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -g\left[\langle\hat{\sigma}_b^{\dagger}\hat{b}\rangle + \langle\hat{b}^{\dagger}\hat{\sigma}_b\rangle\right] - \frac{\Omega}{2}\left[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^{\dagger}\rangle\right] + \gamma\langle\hat{\eta}_b\rangle, \tag{22}$$

where

$$\hat{\eta}_a = |a\rangle\langle a|, \tag{23}$$

$$\hat{\eta}_b = |b\rangle\langle b|, \tag{24}$$

$$\hat{\eta}_c = |c\rangle\langle c|. \tag{25}$$

Equations (17)-(22) are nonlinear differential equations. Now, by applying the large-time approximation [11], the solutions of equations (12) and (13) becomes

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a, \tag{26}$$

and

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b. \tag{27}$$

At steady state, these would obviously be exact relationships. When equations. (26), (27), and their adjoints are introduced, one gets

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = g\left[-\frac{2g}{\kappa}\langle\hat{\eta}_b\hat{\sigma}_a\rangle + \frac{2g}{\kappa}\langle\hat{\eta}_a\hat{\sigma}_a\rangle - \frac{2g}{\kappa}\langle\hat{\sigma}_b^{\dagger}\hat{\sigma}_c\rangle\right] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger}\rangle - \gamma\langle\hat{\sigma}_a\rangle. \tag{28}$$

Considering equations (2), (5), (6) and their adjoints, one obtains

$$\langle\hat{\eta}_b\hat{\sigma}_a\rangle = \langle|b\rangle\langle b|b\rangle\langle a|\rangle = \langle|b\rangle\langle a|\rangle = \langle\hat{\sigma}_a\rangle, \tag{29}$$

$$\langle\hat{\eta}_a\hat{\sigma}_a\rangle = 0, \tag{30}$$

$$\langle\hat{\sigma}_b^{\dagger}\hat{\sigma}_c\rangle = \langle|b\rangle\langle c|c\rangle\langle a|\rangle = \langle|b\rangle\langle a|\rangle = \langle\hat{\sigma}_a\rangle. \tag{31}$$

Substitution of equations (29)-(31) into (28) gives

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -(\gamma + \gamma_c)\langle\hat{\sigma}_a\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger}\rangle. \tag{32}$$

Similarly, the equations of evolution of the atomic operators' expectation values take the form

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{1}{2}(\gamma + \gamma_c)\langle\hat{\sigma}_b\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger}\rangle, \tag{33}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -\frac{1}{2}(\gamma + \gamma_c)\langle\hat{\sigma}_c\rangle + \frac{\Omega}{2}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle], \tag{34}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -(\gamma + \gamma_c)\langle\hat{\eta}_c\rangle + \frac{\Omega}{2}\left[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^{\dagger}\rangle\right], \tag{35}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = -(\gamma+\gamma_c)\langle\hat{\eta}_b\rangle + (\gamma+\gamma_c)\langle\hat{\eta}_a\rangle, \quad (36)$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = (\gamma+\gamma_c)\langle\hat{\eta}_b\rangle - \frac{\Omega}{2}\left[\langle\hat{\sigma}_c^{\dagger}\rangle + \langle\hat{\sigma}_c\rangle\right]. \quad (37)$$

With

$$\gamma_c = \frac{4g^2}{\kappa}, \quad (38)$$

is the stimulated emission decay constant. The completeness relation has the form [12]

$$\hat{\eta}_a + \hat{\eta}_b + \hat{\eta}_c = \hat{I}. \quad (39)$$

Then, we see that [13, 14]

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1, \quad (40)$$

where $\langle\hat{\eta}_a\rangle$ is the probability to find the atom in the top level, $\langle\hat{\eta}_b\rangle$ is the probability to find the atom in intermediate level, and $\langle\hat{\eta}_c\rangle$ is the probability to find the atom in the bottom level. The steady state solutions of equations (32)-(37) are found to be

$$\langle\hat{\sigma}_a\rangle = \frac{\Omega}{(\gamma+\gamma_c)}\langle\hat{\sigma}_b^{\dagger}\rangle, \quad (41)$$

$$\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{(\gamma+\gamma_c)}\langle\hat{\sigma}_a^{\dagger}\rangle, \quad (42)$$

$$\langle\hat{\sigma}_c\rangle = \frac{\Omega}{(\gamma+\gamma_c)}\left(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle\right), \quad (43)$$

$$\langle\hat{\eta}_a\rangle = \frac{\Omega}{2(\gamma+\gamma_c)}\left(\langle\hat{\sigma}_c^{\dagger}\rangle + \langle\hat{\sigma}_c\rangle\right), \quad (44)$$

$$\langle\hat{\eta}_b\rangle = \langle\hat{\eta}_a\rangle. \quad (45)$$

Furthermore, with the aid of equation. (40), one readily obtains

$$\langle\hat{\eta}_c\rangle = 1 - \langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle. \quad (46)$$

In view of equations (45), equation (46) has the form

$$\langle\hat{\eta}_c\rangle = 1 - 2\langle\hat{\eta}_a\rangle. \quad (47)$$

Now, on account of equation (47), equation (43) can be expressed as

$$\langle\hat{\sigma}_c\rangle = \frac{\Omega}{(\gamma+\gamma_c)}\left[1 - 3\langle\hat{\eta}_a\rangle\right]. \quad (48)$$

With the aid of equation (48), one can observe that

$$\langle\hat{\sigma}_c^{\dagger}\rangle = \langle\hat{\sigma}_c\rangle. \quad (49)$$

Also, from equations (41), (42), and (43), one can readily obtains

$$\langle\hat{\sigma}_a\rangle = \langle\hat{\sigma}_b\rangle = 0. \quad (50)$$

$$\langle\hat{\eta}_a\rangle = \frac{\Omega}{(\gamma+\gamma_c)}\langle\hat{\sigma}_c\rangle. \quad (51)$$

By substituting equation (48) into (51) yields

$$\langle\hat{\eta}_a\rangle = \frac{\Omega^2}{(\gamma+\gamma_c)^2+3\Omega^2}. \tag{52}$$

Moreover, on account of equation (45), one can obtain

$$\langle\hat{\eta}_b\rangle = \frac{\Omega^2}{(\gamma+\gamma_c)^2+3\Omega^2}. \tag{53}$$

Now, by Substituting (52) in (47), we have

$$\langle\hat{\eta}_c\rangle = \frac{\Omega^2+(\gamma+\gamma_c)^2}{(\gamma+\gamma_c)^2+3\Omega^2}. \tag{54}$$

Finally, on account of equation (52), equation (48) takes the form

$$\langle\hat{\sigma}_c\rangle = \frac{\Omega(\gamma+\gamma_c)}{(\gamma+\gamma_c)^2+3\Omega^2}. \tag{55}$$

3. Photon Statistics

The mean photon number for the cavity light modes a and b is given by [15]

$$\bar{n}_a = \langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle. \tag{56}$$

On account of equations (26) and (52), equation (56) can be written as

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{(\gamma+\gamma_c)^2+3\Omega^2}\right]. \tag{57}$$

For non-spontaneous case ($\gamma = 0$), the mean photon number of light mode a has the form

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\left(\frac{\Omega^2}{\gamma_c^2+3\Omega^2}\right). \tag{58}$$

In addition, for $\Omega \gg \gamma_c$, equation (58) becomes

$$\bar{n}_a = \frac{\gamma_c}{3\kappa}. \tag{59}$$

The mean photon number of light mode b is determined using the same procedure as

$$\bar{n}_b = \langle\hat{b}^\dagger\hat{b}\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle = \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{(\gamma+\gamma_c)^2+3\Omega^2}\right]. \tag{60}$$

For non-spontaneous case, equation (60) takes the form

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left(\frac{\Omega^2}{\gamma_c^2+3\Omega^2}\right). \tag{61}$$

In addition, for $\Omega \gg \gamma_c$, equation (61) reduces to

$$\bar{n}_b = \frac{\gamma_c}{3\kappa}. \tag{62}$$

The mean photon number for light modes **a** and **b** is the same in both spontaneous and non-spontaneous scenarios, as shown above. The mean photon number for two-mode cavity light can then be expressed as follows

$$\bar{n}_c = \langle\hat{c}^\dagger\hat{c}\rangle. \tag{63}$$

The mean photon number has the form when using the steady state solution of equation (14) and its adjoint

$$\bar{n}_c = \frac{\gamma_c}{\kappa}(\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle). \tag{64}$$

Substituting equations (57) and (58) in (64) for the steady state solution of (14) yields

$$\bar{n}_c = \frac{\gamma_c}{\kappa}\left(\frac{2\Omega^2}{(\gamma+\gamma_c)^2+3\Omega^2}\right). \tag{65}$$

Now, the mean photon number in the non-spontaneous scenario is in the form

$$\bar{n}_c = \frac{\gamma_c}{\kappa}\left(\frac{2\Omega^2}{\gamma_c^2+3\Omega^2}\right). \tag{66}$$

For $\Omega \gg \gamma_c$, equation (66) reduces to

$$\bar{n}_c = \frac{2\gamma_c}{3\kappa}. \tag{67}$$

Furthermore, the variance of the photon number is expressible as [3, 5]

$$(\Delta n)^2 = \langle\bar{n}^2\rangle - \langle\bar{n}\rangle^2. \tag{68}$$

On account of equation (56), the variance of the photon number for light mode a is described as

$$(\Delta n_a)^2 = \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^{\dagger 2}\rangle\langle\hat{a}^2\rangle. \tag{69}$$

Upon use of equations (26) and (50), one obtains

$$\langle\hat{a}\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}_a\rangle \equiv 0. \tag{70}$$

Moreover,

$$\langle\hat{a}^2\rangle = \langle\left(-\frac{2g}{\kappa}\hat{\sigma}_a\right)\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\sigma}^2\rangle. \tag{71}$$

In view of equation (5), one readily obtains

$$\langle\hat{a}^2\rangle = \frac{\gamma_c}{\kappa}\langle(|b\rangle\langle a|)^2\rangle = 0. \tag{72}$$

Equations (69), and (71) are used to calculate the variance of the photon number for light mode a as

$$(\Delta n_a)^2 = \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\hat{a}^\dagger\rangle. \tag{73}$$

On account of equations (52), (53), and (56), equation (73) becomes

$$(\Delta n_a)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2\left[\frac{\Omega^4}{((\gamma+\gamma_c)^2+3\Omega^2)^2}\right]. \tag{74}$$

Furthermore, for non-spontaneous case, the photon number variance has the form

$$(\Delta n_a)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2\left[\frac{\Omega^4}{\left(\gamma_c^2+3\Omega^2\right)^2}\right]. \tag{75}$$

For $\Omega \gg \gamma_c$,

$$(\Delta n_a)^2 = \left(\frac{\gamma_c}{3\kappa}\right)^2. \quad (76)$$

With the same procedure one can obtain the variance of the photon number for light mode b as

$$(\Delta n_b)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{\Omega^4+\Omega^2(\gamma+\gamma_c)^2}{((\gamma+\gamma_c)^2+3\Omega^2)^2}\right]. \quad (77)$$

For non-spontaneous case, equation (75) takes the form

$$(\Delta n_b)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{\Omega^2(\Omega^2+\gamma_c^2)^2}{(\gamma_c^2+3\Omega^2)^2}\right]. \quad (78)$$

For $\Omega \gg \gamma_c$, equation (75) reduces to

$$(\Delta n_b)^2 = \left(\frac{\gamma_c}{3\kappa}\right)^2 = (\Delta n_b)^2 = \bar{n}_a^2 \equiv \bar{n}_b^2, \quad (79)$$

which represents the normally-ordered variance of the photon number for the chaotic light. Furthermore, equation (79) indicates that $(\Delta n_a)^2 > \bar{n}_a$ and $(\Delta n_b)^2 > \bar{n}_b$ and hence the photon statistics of each light-mode is super-poissonian.

With the same approach one can readily obtain the variance of the photon number for superposed light modes **a** and **b** as

$$(\Delta n_c)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{4\Omega^4+3\Omega^2(\gamma+\gamma_c)^2}{((\gamma+\gamma_c)^2+3\Omega^2)^2}\right]. \quad (80)$$

For non-spontaneous case, equation (80) has the form

$$(\Delta n_c)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{4\Omega^4+3\Omega^2\gamma_c^2}{(\gamma_c^2+3\Omega^2)^2}\right]. \quad (81)$$

Additionally, $\Omega \gg \gamma_c$, equation (81) reduces to

$$(\Delta n_c)^2 = \left(\frac{2\gamma_c}{3\kappa}\right)^2 \equiv \bar{n}_c^2, \quad (82)$$

which represents the normally-ordered variance of the photon number for chaotic light. Furthermore, inspection of equation (82) indicates that $(\Delta n_c)^2 > \bar{n}_c^2$ and hence the photon statistics of the two-mode light is super-poissonian.

4. Quadrature Squeezing and The mean Photon number

The squeezing properties of light mode a are described by the two quadrature operators [16-18]

$$\hat{a}_+ = \hat{a}^+ + \hat{a}, \quad (83)$$

and

$$\hat{a}_- = i(\hat{a}^+ - \hat{a}). \quad (84)$$

In view of equations (83) and (84), the commutation relation becomes

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{\eta}_a - \hat{\eta}_b). \quad (85)$$

The uncertainty relation for the two Hermitian operators $\hat{A}$ and $\hat{B}$ satisfies the commutation relation $i\hat{C}$, which is described as [5]

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle \hat{C} \rangle\right|. \tag{86}$$

Upon use of equation (86), one can readily obtains

$$\Delta a_{+}\Delta a_{-} \geq \frac{1}{2}|\langle[\hat{a}_{-},\hat{a}_{+}]\rangle| \geq |\langle\hat{a}^{+}\hat{a}\rangle - \langle\hat{a}\hat{a}^{+}\rangle|. \tag{87}$$

On account of equation (57) along with (75), one obtains

$$\Delta a_{+}\Delta a_{-} \geq 0. \tag{88}$$

Next the variance of the plus and minus quadrature operators becomes [17]

$$(\Delta a_{+})^2 = \langle \hat{a}_{+}^2 \rangle - \langle \hat{a}_{+} \rangle^2, \tag{89}$$

and

$$(\Delta a_{-})^2 = \langle \hat{a}_{-}^2 \rangle - \langle \hat{a}_{-} \rangle^2. \tag{90}$$

In consideration of equation (84), equation (87) can be expressed in terms of the raising and lowering operators as

$$(\Delta a_{\pm})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle \pm \langle \hat{a} \rangle^2 \pm \langle \hat{a}^{\dagger} \rangle^2 \mp \langle \hat{a}^2 \rangle \mp \langle \hat{a}^{\dagger 2} \rangle \mp 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle. \tag{91}$$

In view of equations (70) and (72), equation (91) reduces to

$$(\Delta a_{\pm})^2 = \langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{a}\hat{a}^{\dagger} \rangle. \tag{92}$$

Now, by using equations (56) and (64), one obtains

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa}[\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle]. \tag{93}$$

On substituting equations (52) and (53), the quadrature variance for the light mode a becomes

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\frac{2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{94}$$

For non-spontaneous case $(\gamma = 0)$, the quadrature variance has the form

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\frac{2\Omega^2}{(\gamma_c)^2+3\Omega^2}\right]. \tag{95}$$

In addition, for $\Omega \gg \gamma_c$, equation (94) reduces to

$$(\Delta a_{\pm})^2 = \frac{2\gamma_c}{3\kappa}. \tag{96}$$

In view of equation (63), the quadrature variance of light mode a can be written in terms of the mean photon number as

$$(\Delta a_{\pm})^2 = 2\bar{n}_a, \tag{97}$$

which is the normally-ordered quadrature variance for chaotic light. In the absence and presence of spontaneous emission the mean photon number of the two-mode light is the same as with the quadrature variance of light mode a. This can be written as

$$\bar{n}_c = (\Delta a_{\pm})^2. \tag{98}$$

In the same procedure the quadrature variance of light mode b can be obtained as

$$(\Delta b_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\frac{2\Omega^2+(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{99}$$

For $\gamma = 0$, equation (99) reduces to

$$(\Delta b_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\frac{2\Omega^2+\gamma_c{}^2}{\gamma_c{}^2+3\Omega^2}\right]. \tag{100}$$

And for $\Omega \gg \gamma_c$, Hence, equation (100) merely becomes

$$(\Delta b_{\pm})^2 = \frac{2\gamma_c}{3\kappa}. \tag{101}$$

Thus, the quadrature variance of light mode b is written in terms of the mean photon number as

$$(\Delta ab_{\pm})^2 = 2\bar{n}_b, \tag{102}$$

which is the normally-ordered quadrature variance for chaotic light. The squeezing properties of the two-mode cavity light can be described as

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{103}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{104}$$

where,

$$\hat{c} = \hat{a} + \hat{b}. \tag{105}$$

With the aid of equations (103) and (104), the commutation relation is found to be

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{\eta}_a - \hat{\eta}_c). \tag{106}$$

The quadrature operators' uncertainty relation for two-mode cavity light is expressed as [10, 11]

$$\Delta c_+ \Delta c_- \geq \frac{1}{2}|\langle[\hat{c}_-, \hat{c}_+]\rangle|. \tag{107}$$

Now, in view of equation (106), one can re-write equation (107) as

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}|\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_c\rangle|. \tag{108}$$

By employing equations (53) and (54), equation (108) can be written as

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\left|\frac{(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right|. \tag{109}$$

In the absence of spontaneous emission $(\gamma = 0)$, it becomes

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\left|\frac{\gamma_c^2}{\gamma_c^2+3\Omega^2}\right|. \tag{110}$$

In the absence of deriving coherent light $(\Omega = 0)$

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}, \tag{111}$$

which is the uncertainty relation for vacuum state. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined as [19]

$$(\Delta c_+)^2 = \langle \hat{c}_+^2 \rangle - \langle \hat{c}_- \rangle^2, \tag{112}$$

and

$$(\Delta c_-)^2 = \langle \hat{c}_-^2 \rangle - \langle \hat{c}_- \rangle^2. \tag{113}$$

On account of equations (105), (112) and (113), the plus and minus quadrature variance for the creation and annihilation operators can be written as

$$(\Delta c_\pm)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle \pm \langle \hat{c} \rangle^2 \pm \langle \hat{c}^\dagger \rangle^2 \mp \langle \hat{c}^2 \rangle \mp \langle \hat{c}^{\dagger 2} \rangle \mp 2\langle \hat{c} \rangle \langle \hat{c}^\dagger \rangle. \tag{114}$$

Now, using the steady state solution of equation (14) along with (50), one can get

$$\langle \hat{c} \rangle = 0. \tag{115}$$

In view of equation (115), the quadrature variance becomes

$$(\Delta c_\pm)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle \pm \langle \hat{c} \rangle^2 \pm \langle \hat{c}^\dagger \rangle^2. \tag{116}$$

Thus, with the aid of equation (68) along with (111), equation (116) becomes

$$(\Delta c_\pm)^2 = \frac{\gamma_c}{\kappa}\left[\langle \hat{\eta}_a \rangle + 2\langle \hat{\eta}_b \rangle \pm \langle \hat{\eta}_c \rangle \pm \langle \hat{\sigma}_c \rangle \pm \langle \hat{\sigma}_c^\dagger \rangle\right]. \tag{117}$$

By substituting equations (64)-(57) in (117), one obtains

$$(\Delta c_\pm)^2 = \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2 + (\gamma_c+\gamma)^2 \pm 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]. \tag{118}$$

In the absence of spontaneous emission ($\gamma = 0$), equation (118) turns to

$$(\Delta c_\pm)^2 = \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2 + \gamma_c^2 \pm 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]. \tag{119}$$

Besides, for $\Omega \gg \gamma_c$, equation (119) will have the form

$$(\Delta c_\pm)^2 = \frac{4\gamma_c}{3\kappa}. \tag{120}$$

In view of equation (71), equation (120) can be expressed as

$$(\Delta c_\pm)^2 = 2\bar{n}_c, \tag{121}$$

where this represents the normally-ordered quadrature variance for chaotic light. For $\Omega = 0$, equations (69), (110), and (119) become

$$\bar{n}_c = 0, \tag{122}$$

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}, \tag{123}$$

$$(\Delta c_\pm)^2 = \frac{\gamma_c}{\kappa}. \tag{124}$$

The mean photon number and quadrature variance of a two-mode vacuum condition are represented by equations (122), (123), and (124).

The quadrature squeezing of two-mode cavity light in relation to the quadrature variance of the two-mode cavity vacuum state can be determined using the formula [20]

$$S = \frac{(\Delta c_-)_v^2 - (\Delta c_-)^2}{(\Delta c_-)_v^2}. \tag{125}$$

Equations (109) and (120) are used to obtain

$$S = 1 - \left[\frac{4\Omega^2+(\gamma_c+\gamma)^2-2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2+3\Omega^2}\right] \equiv \frac{2\Omega(\gamma_c+\gamma)-\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}. \quad (126)$$

For $\gamma = 0,$ the above expression reduces to

$$S = \frac{2\Omega\gamma_c-\Omega^2}{\gamma_c^2+3\Omega^2} \equiv \frac{2\eta-\eta^2}{3\eta^2+1} \quad \text{with} \quad \eta = \frac{\Omega}{\gamma_c}. \quad (127)$$

5. Physical Analysis

1. Plots mean photon number (As shown in the Figure 2 below)

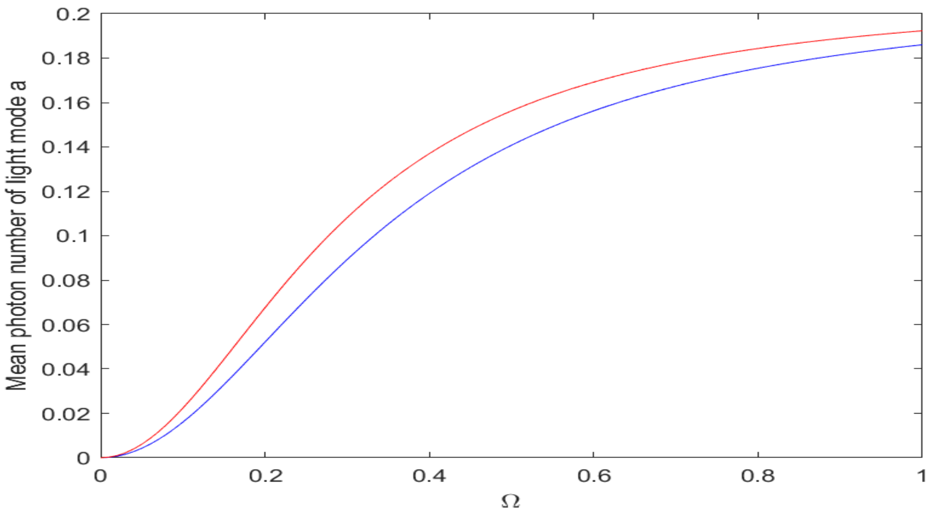

Figure 2. Plots of eqs (57), (58) versus Ω for γ_c=0.5, κ=0.8, γ=0 (red curve) and γ=0.1 (blue curve).

2. Plots of the variance of the photon number for light mode a (As shown in the Figure 3 below)

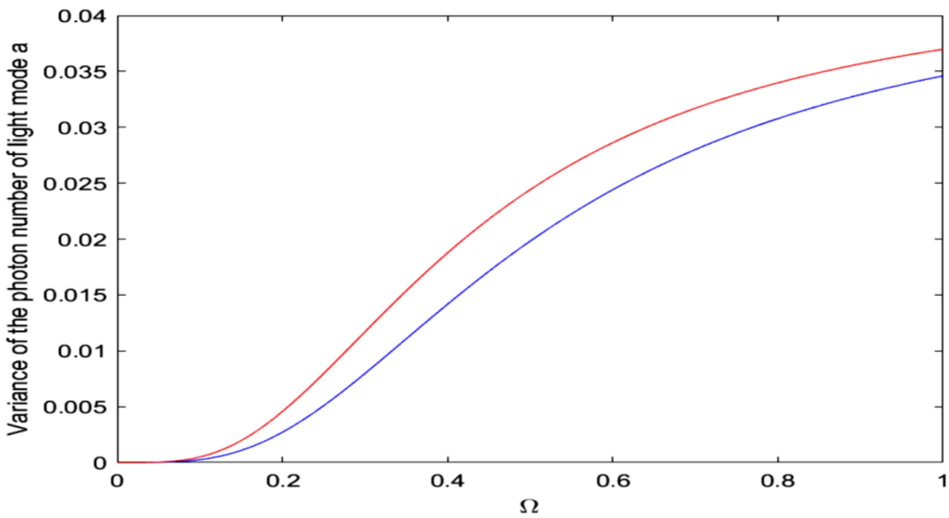

Figure 3. Plots of Eqs. (74) And (75) versus Ω for γ_c=0.5, κ=0.8, γ=0 (red curve) and γ=0.1 (blue curve).

3. Plots of the variance of the photon number for light mode b (As shown in the Figure 4 below)

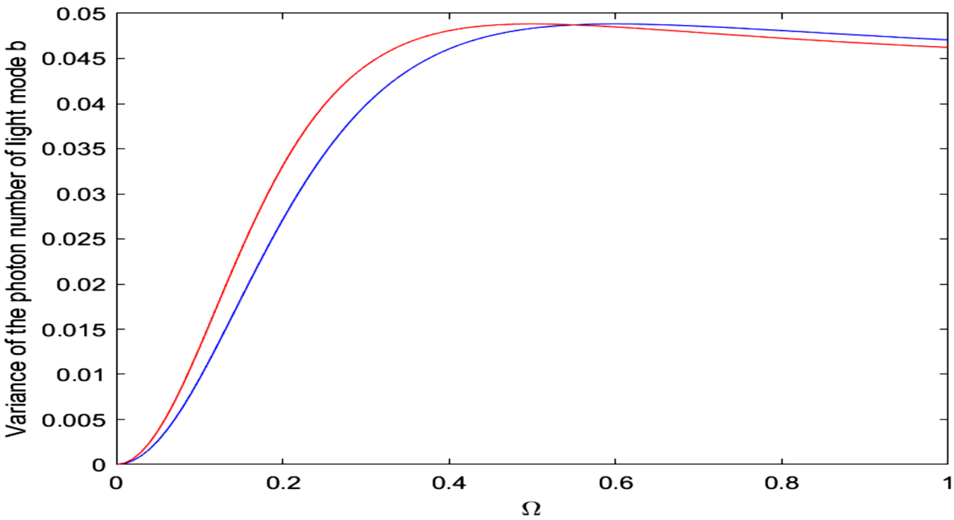

Figure 4. Plots of Eqs. (77) And (78) versus Ω for γ_c=0.5, κ=0.8, γ=0 (red curve) and γ=0.1 (blue curve).

4. Plots of quadrature variance of two modes light (As shown in the Figure 5 below)

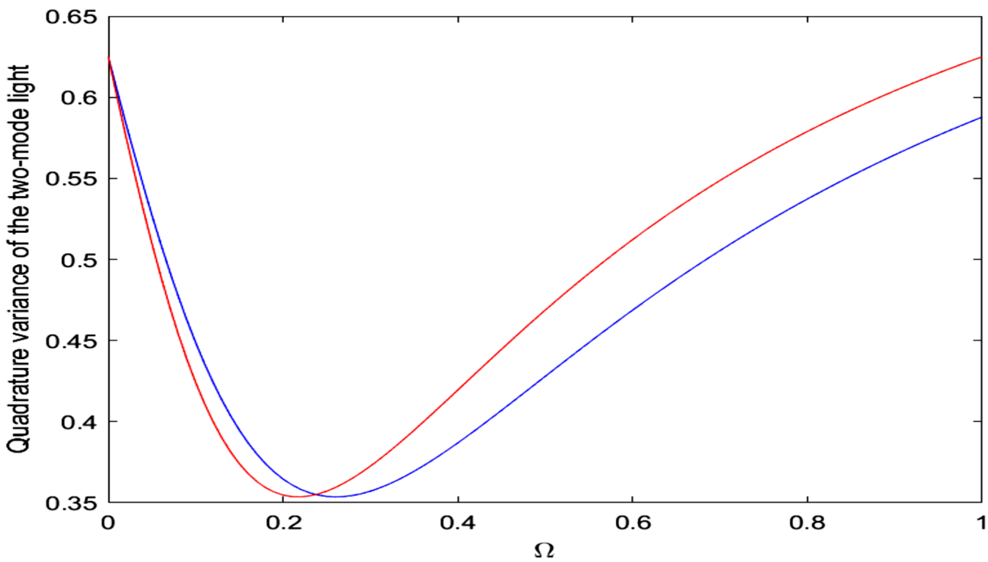

Figure 5. Plots of Eqs. (118) and (119) versus Ω for γ_c=0.5, γ=0 & κ=0.8 (red color) and γ=0.1 (blue color).

5. Plots of quadrature squeezing (As shown in the Figure 6 below)

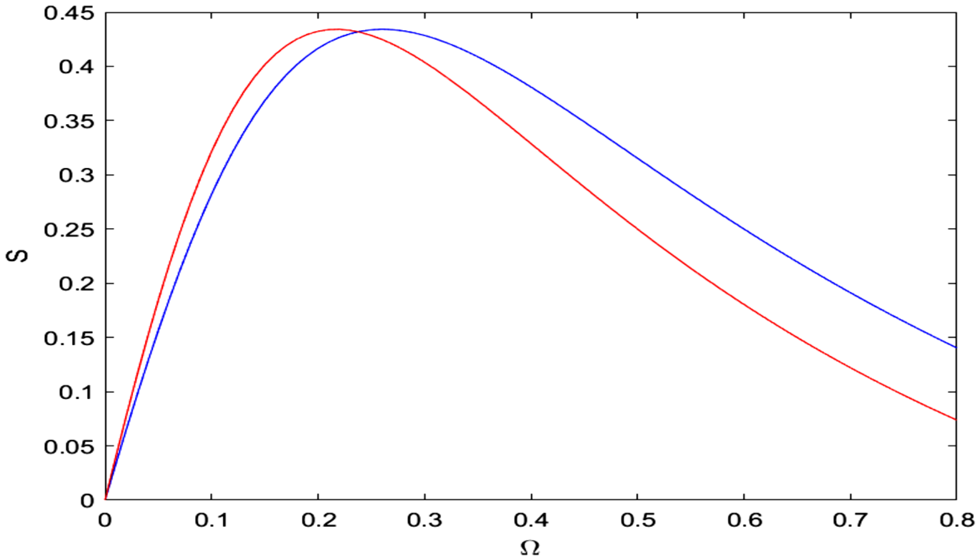

Figure 6. Plots of Eq. (125) and (126) versus Ω for γ_c=0.5, γ=0 (red color) and γ=0.1 (blue color).

According to Figs. 1 and 2, light mode a's mean photon number and photon number variance are higher than those for $\gamma = 0$. The plots, however, overlap at that moment $\Omega = 0.55$ in Figure 3. This demonstrates that when the variation of the light's photon number is greater for $\gamma = 0$ mode b than for $\gamma = 0.1$, and vice versa.

The plot from Fig.4 clearly demonstrates that the quadrature variance of the two-mode light is less in the absence of spontaneous emission when $\Omega < 0.24$ and is anticipated to be bigger in the absence of spontaneous emission when $\Omega > 0.24$ for the quadrature variance of two light modes. Finally, we discovered from Figure 5 that the maximum quadrature squeezing for both light modes is 43.42 percent and that the plots intersect at the spot.

6. Conclusion

A coherently driven three-level atom with an open cavity coupled to a two-mode vacuum reservoir by a single port mirror has its quantum features thoroughly examined. The master equation was used to find the steady-state solutions of the equations of motion for linear operators and the equation of evolution of the expectation values of atomic operators with stable solutions. Using steady state solutions of the equations of motion for linear operators and equations of evolution of the expectation values, we estimated the mean photon number, the photon number variance, and the quadrature variance for single-mode cavity light beams as well as two-mode light beams. We also calculated quadrature squeezing for the two mode-lights. The mean photon number, the variance of the photon number for light mode a, the variance of the photon number for the two-mode cavity light, and the quadrature variance of light mode a for γ = 0 is greater than for γ = 0.1. From the plots of variance of the photon number of light mode b cross each other at the point Ω=0.55. This shows that when Ω<0.55 the variance of the photon number for γ=0 is greater than for γ=0.1 and vice versa. From the calculation the quadrature variance of light mode b for γ=0 is less than for γ=0.1. The quadrature variance of the two-mode cavity light is less in the absence of spontaneous emission when Ω<0.24 and grater in the absence of luminescence when Ω>0.24. The plots of quadrature squeezing cross each other at the point Ω=0.24. When Ω<0.24, the quadrature squeezing for γ=0 is greater than that for γ=0.1

and vice versa. Finally, it was found that the maximum quadrature squeezing of the two-mode cavity light is 43.42% for both γ=0 and γ=0.1 below the vacuum-state level.

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