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Article

Role of Skew-Symmetric Differential Forms in Mathematical Physics and Field Theory

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Abstract: Skew-symmetric differential forms possess properties that enable one to carry out a qualitative investigation of the equations of mathematical physics and the foundations of field theories. In the paper we call attention to a unique role in field theory of closed exterior skew-symmetric differential forms, which correspond to conservation laws for physical fields (to conservative quantities). At the same time, it was shown that such closed exterior forms can be derived from skew-symmetric differential forms, which follow from the mathematical physics equations describing material media such as thermodynamic, gas-dynamic, cosmic media. This points a connection the field theory equations with the mathematical physics equations. Such connection discloses the properties and specific features of field theory.

Keywords: Mathematical Physics Equations; Field Theory Equations; Conservation Laws; Skew-Symmetric Differential Forms; Material Media

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1. Introduction

The physical meaning of skew-symmetric differential forms is connected with the fact that they correspond to conservation laws that are the basis equations of mathematical physics and field theory.

Closed exterior forms correspond to conservation laws for physical fields.

In section 2 the properties and specific features of closed exterior differential forms are briefly described, and the role of closed exterior differential forms in field theory shown.

In section 3 shown role of evolutionary skew-symmetric differential forms in mathematical physics and field theory.

It is shown that the closed exterior forms, which describe the conservation laws for physical fields, are obtained from the evolutionary skew-symmetric differential forms that follow from the mathematical physics equations describing material media (such as thermodynamic, gas-dynamic, cosmic media), and which consist of equations of conservation laws for material media.

Realization of closed exterior forms, which describe the conservation laws for physical fields, from evolutionary forms that are obtained from the equations of conservation laws for material media, reveals a connection the field-theory equations with the equations for material media.

2. Role of closed exterior differential forms in field theory

In paper [1] it has been shown the role of skew-symmetric differential forms in mathematics. The properties of closed exterior and dual forms, namely, invariance, covariance, conjugacy, and duality, either explicitly or implicitly appear in all invariant mathematical formalisms, such as algebra, geometry, differential geometry, vectorial and tensor calculus, the theory of complex variables, and so on.

In this section the properties and specific features of closed exterior differential forms will be briefly described, and the role of closed exterior differential forms in field theory will be shown.

2.1. Some specific features of closed exterior differential forms

The exterior differential form of degree p (p -form) can be written as (see [2-5])

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \quad 0 \le p \le n \tag{1}$$

Here $a_{i_1 \dots i_p}$ are the functions of the variables x^{i_1}, x^{i_2}, \dots, x^{i_n}, n is the dimension of the space, $\wedge$ is the operator of exterior multiplication, dx^i, $dx^i \wedge dx^j$, $dx^i \wedge dx^j \wedge dx^k$, \dots\ is the local basis which satisfies the condition of exterior multiplication:

$$\begin{aligned} &dx^i \wedge dx^i = 0 \\ &dx^i \wedge dx^j = -dx^j \wedge dx^i \quad i \ne j \end{aligned} \tag{2}$$

Below, the symbol of summing, $\sum$, and the symbol for exterior multiplication, $\wedge$, will be omitted. The summing over repeated indices is implied.

The differential of the exterior form θ^p is expressed as

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} dx^{i_1} dx^{i_2} \dots dx^{i_p} \tag{3}$$

At this point it should call attention to the fact that in the differential of exterior form the term with the basis differential absent. This is due to that the exterior form is defined on integrable manifold. In contrast to this, the skew-symmetric, which, as it will be shown below, generate closed exterior forms, are defined on nonintegrable manifold.

An exterior form is called 'closed' if its differential is equal to zero:

$$d\theta^p = 0 \tag{4}$$

From condition (4) one can see that the closed exterior form is a conserved quantity. This means that it corresponds to a conservation law, namely, to some conserved physical quantity.

A form which is the differential of some other form:

$$\theta^p = d\theta^{p-1} \tag{5}$$

is called an 'exact' form. Exact forms prove to be closed automatically: $d\theta^p = dd\theta^{p-1} = 0$.

As it will be shown below, the most application in mathematical physics and field theory have closed inexact exterior forms that are defined only on a certain structure. The structure, on which the exterior differential form can become a closed inexact form, is a pseudostructure with respect to its metric properties. Cohomology, sections of cotangent bundles, integral and potential surfaces, eikonal surfaces and so on, may be regarded as examples of pseudostructures or manifolds made up by pseudostructures.

If the form is closed on pseudostructure only, the closure condition is written as

$$d_{\pi}\theta^{p} = 0 \tag{6}$$

And the pseudostructure π obeys the condition

$$d_{\pi}{}^{*}\theta^{p} = 0 \tag{7}$$

where ${}^{*}\theta^{p}$ is a dual form.

From conditions (6) and (7) one can see that the closed inexact exterior form and relevant dual form made up a differential-geometrical structure that describes a conserved object, namely, a pseudostructure with a conservative quantity. Such conservative objects can also correspond to some conservation laws.

The closed inexact exterior form is an interior (i.e. defined only on pseudostructure) differential, that is

$$\theta^{p}_{\pi} = d_{\pi}\theta^{p-1} \tag{8}$$

From this it follows that the form of lower degree can correspond to a potential, and the closed form by itself can correspond to a potential force. This is an additional example showing that a closed form may have physical meaning. Here the two-fold nature of a closed form is revealed, on the one hand, as a locally conserved quantity, and on the other hand, as a potential force.

Closed exterior and dual forms possess the properties (such as an invariance, covariance, conjugacy, and duality) that manifest themselves practically in all formalisms of field theory.

2.1.1. Invariance

Since exact and inexact closed forms are differentials, it is obvious that these closed forms will appear to be invariant under all transformations that conserve the differential. Transformations conserving the differential lie at the basis of many branches of mathematics, mathematical physics and field theory. The nondegenerate transformations, such as the unitary, tangent, canonical, gradient, and other nondegenerate transformations are examples of transformations that conserve the differential. These are the gauge transformations for spinor, scalar, vector, and tensor fields.

A large number of operators of mathematics and mathematical physics are connected with nondegenerate transformations of exterior differential forms. The exterior differential operator, d, is an abstract generalization of the ordinary mathematical operations of gradient, curl and divergence in vector calculus. If, in addition to the exterior differential, we introduce the following operators: (1) δ for transformations that convert the form of degree $p+1$ into the form of degree p, (2) δ' for cotangent transformation, (3) Δ for the $d\delta - \delta d$ transformation, (4) Δ' for the $d\delta' - \delta' d$ transformation, then in terms of these operators, which act on the exterior differential forms, one can write down the operators in field theory equations. The operator δ corresponds to Green's operator, δ' to the canonical transformation operator, Δ to the d'Alembert operator in 4-dimensional space, and Δ' corresponds to the Laplace operator [6].

The covariance of a dual form is directly connected with the invariance of an exterior closed form.

2.1.2. Invariant structures

As it has been pointed out, the closed inexact exterior form and relevant dual form made up a differential-geometrical structure.

It is evident that such differential-geometrical structures also remain invariant under all transformations that conserve the differential.

Below it will be demonstrated a unique role of differential-geometrical structures in mathematical physics and field theory. The physical structures, which made up physical fields, are described by such differential-geometrical structures.

2.1.3. Invariance as the result of conjugacy of elements of exterior or dual forms

The closure property of an exterior form implies that any objects, namely, elements of the exterior form, components of elements, elements of the form differential, exterior and dual forms and others, turn out to be conjugated. It is a conjugacy that leads to realization of invariant and covariant properties of the exterior and dual forms. A variety of conjugate objects leads to the fact that closed forms can describe a great number of various physical objects and spatial structures.

The conjugacy is connected with another specific property of exterior differential forms, namely, their duality; and, as will be shown later, the duality has a fundamental physical meaning.

2.1.4. Identical relations of closed exterior forms

As it will be shown below, the identical relations of closed exterior forms, which occur practically in all branches of physics, mechanics, thermodynamics and so on, play un unique role in mathematical physics and field theory.

The identical relations is a connection between two operators or mathematical objects, and this is a mathematical expression of conjugace or duality of closed exterior forms.

One has to distinguish several types of identical relations of exterior differential forms.

1) *Relations between differential forms.*

Examples of such identical relations are:

a) the Poincare invariant $ds = -H\,dt + p_j\,dq_j$,

b) the second principle of thermodynamics $dS = (dE + pdV)/T$.

In general, such an identical relation can be written as

$$d\phi = \theta^p \tag{9}$$

In this relation the form in the right-hand side has to be a closed one. As will be shown below, the identical relations, which are satisfied only on pseudostructures, have a physical meaning. Such an identical relation can be written as

$$d_\pi \phi = \theta^p_\pi \tag{10}$$

In identical relations (9) and (10), on the one hand, there is a closed form, and on the other, there is a differential of some differential form of the degree less by one, which is closed external form too.

The relations, in which the skew-symmetric forms are expressed in terms of their analogs, are another identical relations of closed exterior forms. These are the relations:

1. *integral ones* (formulas of Newton, Leibnitz, Green, Stokes, Gauss-Ostrogradskii),
2. *vector ones* (that relate the operators of gradient, curl, divergence and so on),
3. *tensor ones* (such as the gauge relations in electromagnetic field theory, the Bianchi identity in the gravitation theory and so on),

4. *identical relations between derivatives* (such as the Cauchi-Riemann conditions, the transversality condition, the canonical relations in Hamiltonian formalism, the thermodynamic relations and so on).

Below the meaning of identical relations and their importance for the theories, which describe physical fields, will be disclosed.

2.2. Closed exterior forms as the basis of field theories

It is well known that all existing field theories are based on the postulates of invariance or covariance.

The invariance of existing field theories is an invariance of closed exterior forms, i.e. an invariance of closed exterior forms under all transformations that conserve the differential.

The gauge transformations of field theories are precisely such transformations of closed exterior forms. These are unitary transformations (0-form), gauge and canonical transformations (1-form), gradient transformations (2-form) and so on.

The gauge symmetries of field theories are those for closed exterior forms. And the interior symmetries of field theories are those of closed exterior forms, whereas the exterior symmetries of field theory equations are symmetries of dual forms.

The field theory operators relate to nondegenerate transformations of closed exterior differential forms.

Closed inexact exterior or dual forms are solutions of the field-theory equations. And there is the following correspondence:

- Closed exterior forms of zero degree correspond to quantum mechanics.
- The Hamilton formalism bases on the properties of closed exterior and dual forms of first degree.
- The properties of closed exterior and dual forms of second degree are at the basis of the equations of electromagnetic field.
- The closure conditions of exterior and dual forms of third degree form the basis of equations for gravitational field.

One can see that field theories equations, as well as the gauge transformations in field theories, are connected with closed exterior forms of a certain degree. This enables one to introduce a classification of physical fields and interactions in degrees of closed exterior forms. If to denote the degree of closed exterior forms by k, then $k=0$ corresponds to strong interaction, $k=1$ does to weak one, $k=2$ does to electromagnetic one, and $k=3$ corresponds to gravitational interaction.

Such a classification shows that there exists an internal connection between field theories that describe physical fields of various types. It is evident that the degree of closed exterior forms is a parameter that integrates field theories into unified field theory. This can serve as a step to constructing the unified field theory.

The connection of field theory and the theory of closed exterior forms is explained by the fact that the closure conditions of exterior and dual forms, i.e. the conditions of invariance and covariance, correspond to conservation laws which physical fields are subject to.

One can see that the existing invariant field theories are based on the properties of closed exterior forms.

However, the theory of closed exterior forms cannot be completed without an answer to a question of how the closed exterior forms emerge.

It will be shown below that the closed exterior forms, which describe the conservation laws for physical fields, are obtained from the skew-symmetric differential forms, which are evolutionary ones. They can be derived from the mathematical physics equations describing material media such as thermodynamic, gas-dynamic, cosmic media. The properties of such skew-symmetric forms and manifolds on which they are defined are outlined in paper [7].

In present paper all necessary information will be reviewed in further presentation.

3. Role of evolutionary skew-symmetric differential forms in mathematical physics and field theory

Role of evolutionary skew-symmetric differential forms in mathematical physics and field theory relates to the fact that they generate closed exterior forms whose properties lie at the basis of field theories. The process of obtaining closed exterior forms from evolutionary forms discloses the mechanism of evolutionary processes which proceed in material media (systems) and are accompanied by origination of physical structures, which made up physical fields. This points (as will be shown below) out to the connection between physical fields and material media and demonstrate the connection between the field-theory equations describing physical fields and the mathematical physics equations describing material media.

The skew-symmetric forms, which can generate closed exterior forms, are defined on nonintegrable manifolds in contrast to exterior forms. Therefore, they differ in its properties from exterior forms, which are defined on integrable manifolds or structures.

3.1. Some properties of skew-symmetric forms with the basis on nonintegrable manifolds

Examples of nonintegrable manifolds are the tangent manifolds of differential equations that describe an arbitrary processes, Lagrangian manifolds, and the manifolds made up by trajectories of material medium elements (particles), which are obtained while describing evolutionary processes in material media.

3.1.1. Some properties of manifolds

Assume that on a manifold one can introduce a coordinate system with base vectors $\mathbf{e}_\mu$ and define the metric forms for a manifold [8]: $(\mathbf{e}_\mu \mathbf{e}_\nu)$, $(\mathbf{e}_\mu dx^\mu)$, $(d\mathbf{e}_\mu)$.

If the metric form is closed (i.e., its commutators equal zero), this metric is defined by $g_{\mu\nu} = (\mathbf{e}_\mu \mathbf{e}_\nu)$ and the results of translation of the point $d\mathbf{M} = (\mathbf{e}_\mu dx^\mu)$ and the unit frame $d\mathbf{A} = (d\mathbf{e}_\mu)$ over a manifold prove to be independent of the path of integration. Such a manifold is integrable one. (On the integrability of manifolds, see [8]).

If metric forms are unclosed (the commutators of metric forms are nonzero), this points to the fact that this manifold is nonintegrable.

Metric forms and their commutators define the metric and differential characteristic of the manifold.

Closed metric forms define the manifold structure, i.e. the internal characteristic of the manifold. And, unclosed metric forms define the differential characteristic of the manifold. The topological properties of manifolds are connected with commutators of unclosed metric forms. The commutators of unclosed metric forms define the manifold differential characteristic that specify the manifold deformations: bending, torsion, rotation, twist. Thus, the final result is that nonintegrable manifolds, i.e. the manifolds with unclosed metric forms, are deformed manifolds.

Since nonintegrable manifolds are deformed manifolds, it is evident that the skew-symmertic forms defined on such manifolds are evolutionary ones.

The properties of evolutionary forms (which generate closed exterior forms) and specific features of their mathematical apparatus relate to peculiarities of the evolutionary form differential, which depends on the basic differential (i.e. on the properties of nonintegrable manifolds).

3.1.2. Peculiarities of the evolutionary form differential

The evolutionary differential form of degree p can be written similarly to the exterior differential form.

However, the differential of the evolutionary form cannot be written in a manner similar to that described above for the exterior differential forms (see formula (3)). In the evolutionary form differential there appears an additional term being connected with the fact that the basis of the form changes, since the manifold, on which the evolutionary form is defined, is nonintegrable one.

The evolutionary form differential can be written as [1]

$$d\omega^p = \sum_{\alpha_1\ldots\alpha_p} da_{\alpha_1\ldots\alpha_p} dx^{\alpha_1} dx^{\alpha_2}\ldots dx^{\alpha_p} + \sum_{\alpha_1\ldots\alpha_p} a_{\alpha_1\ldots\alpha_p} d(dx^{\alpha_1} dx^{\alpha_2}\ldots dx^{\alpha_p}) \qquad (11)$$

where the second term is connected with the differential of the basis.

For differential forms defined on a manifold with unclosed metric form, one has $d(dx^{\alpha_1}dx^{\alpha_2}\ldots dx^{\alpha_p}) \neq 0$. And for a manifold with a closed metric form, the following relationship $d(dx^{\alpha_1}dx^{\alpha_2}\ldots dx^{\alpha_p}) = 0$ is valid.

The second term in the expression for the differential of skew-symmetric form connected with the differential of the basis is expressed in terms of the metric form commutator [1].

3.1.3. Nonclosure of evolutionary differential forms

Since metric forms of the manifold are unclosed, the second term of the differential form commutator includes the metric form commutator that is not equal to zero. In addition, the terms of the evolutionary form commutator have a different nature. Such terms cannot compensate one another. For this reason, the differential form commutator proves to be nonzero. And this means that the differential form defined on the manifold with an unclosed metric form cannot be closed.

As it will be shown below, the mathematical apparatus of skew-symmetric forms defined on nonintegrable manifolds that are evolutionary unclosed forms, includes inconveniened elements, such as degenerate transformations, nonidentical relations and transition from nonintegrable manifold to integrable one. (This is outlined in details in paper [1].) Such peculiarities of the mathematical apparatus of evolutionary differential forms enable one to disclose the mechanism of generation of closed exterior forms.

3.2. Mechanism of generation of closed exterior forms corresponding to the conservation laws for physical fields

As it was already noted, existing field theories are based on the properties of closed exterior forms, which correspond to the conservation laws for physical fields. And it will be shown that they are realized from skew-symmetric forms, which also correspond to conservation laws. Therefore we have focussed our attention on the concept of "conservation laws".

3.2.1. Conservation laws

Historically it happen so that in the branches of physics connected with field theory (describing physical fields) and in physics of material media (material systems) the concept of "conservation laws" has a different meaning. [As examples of material media it may be the thermodynamic, gas dynamical and cosmic media, the systems of elementary particles and others.]

In mechanics and physics of material media the concept of "conservation laws" relates to the conservation laws for energy, linear momentum, angular momentum, and mass that establish the balance between the change of physical quantities and the external action. These conservation laws can be named "the balance conservation laws". The balance conservation laws are described by differential (or integral) equations [10,11].

In field theory "conservation laws" are those that claim an existence of conservative physical quantities or objects. Such conservation laws are described by closed exterior skew-symmetric forms. The Noether theorem is an example. These are conservation laws for physical fields. (For the sake of convenience the conservation laws for physical fields will be called as "exact".)

It turns out that the balance and exact conservation laws are mutually connected. Closed exterior forms, which describe the conservation laws for physical fields, are obtained from evolutionary forms that in turn are derived from the mathematical physics equations, consisting of the equations of balance conservation laws for material media.

This follows from the analysis of the equations of balance conservation laws for material media.

The analysis of properties and peculiarities of the equations of balance conservation laws for material media enables one to understand the mechanism of generation of closed exterior forms, which correspond to the conservation laws for physical fields and form the basis of field theories.

3.2.2. Peculiarities of the equations of balance conservation laws for material media.

Equations of mathematical physics, which consist of equations of balance conservation laws, describe not only the state of material media. They control evolutionary processes in material media (material systems) that are accompanied by an emergence of physical structures.

And this is due to the properties of the equations of balance conservation laws.

The functional properties and specific features of equations or sets of equations depend on whether or not the derivatives of differential equations or the equation in the sets of differential equations are consistent.

The equations are consistent if they can be contracted into identical relations for the differentials, i.e. for closed forms.

Let us analyze the equations that describe the conservation laws for energy and linear momentum. The energy equation in the inertial frame of reference can be reduced to the form:

$$\frac{D\psi}{Dt} = A_1 \tag{12}$$

where D/Dt is the total derivative with respect to time, ψ is the functional of the state that specifies material system, A_1 is the quantity that depends on specific features of material system and on external energy actions onto material system. The action functional, entropy and wave function can be regarded as examples of the functional ψ. Thus, the equation for energy presented in terms of the action functional S has a similar form: $DS/Dt=L$, where $\psi=S$, $A_1=L$ is the Lagrange function. In mechanics of continuous media the equation for energy of ideal gas can be presented in the form [10]: $Ds/Dt=0$, where s is the entropy. In this case, $\psi=s$, $A_1=0$. It is worth noting that the examples presented show that the action functional and entropy play the same role. Specific features and properties of such functionals are described in paper [12].

In the accompanying frame of reference (this frame of reference is connected with the manifold made up by the trajectories of the material system elements) the total derivative with respect to time is transformed into the derivative along the trajectory. Equation (12) is now written in the form

$$\frac{\partial \psi}{\partial \xi^1}=A_1 \qquad (13)$$

here ξ^1 is the coordinate along the trajectory.

In a similar manner, in the accompanying frame of reference the equation for linear momentum appears to be reduced to the equation of the form:

$$\frac{\partial \psi}{\partial \xi^\nu}=A_\nu, \quad \nu=2,... \qquad (14)$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on specific features of material system and external force actions onto the system.

Eqs. (13), (14) can be convoluted into the relation

$$d\psi=A_\mu d\xi^\mu, \quad (\mu=1,\nu) \qquad (15)$$

where $d\psi$ is the differential expression $d\psi=(\partial\psi / \partial\xi^\mu)d\xi^\mu$.

Relation (15) can be written as

$$d\psi=\omega \qquad (16)$$

here $\omega=A_\mu d\xi^\mu$ is a skew-symmetric differential form of the first degree.

Since the equation of the balance conservation laws are evolutionary ones, skew-symmetric differential form ω and the relation obtained (16) are also evolutionary ones.

Relation (16) was obtained from the equations of the balance conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form in the evolutionary relation will be a form of the second degree. And in combination with the equation of the balance conservation law of mass this form will be a form of degree 3.

Thus, in the general case the evolutionary relation can be written as

$$d\psi=\omega^p \qquad (17)$$

where the form degree p takes the values $p=0,1,2,3$.. (The evolutionary relation for $p=0$ is similar to that in the differential forms, and it was obtained from the interaction of energy and time.)

Relations (16, 17), obtained from the equations of balance conservation laws for material systems, are describing the process of generation of closed (inexact) exterior form. This appears to be possible due to peculiarities that this relation possesses. This relation proves to be a nonidentical self-varying relation.

3.2.3. Nonidentity of evolutionary relation

Relation can be identical one if it contains only invariant measurable terms.

Relation (17) obtained from the equation of balance conservation laws prove to be nonidentical since differential form ω^p, which includes in the right-hand side of relation (17), is unclosed form, and, therefore, it is not an invariant.

Let us consider this by the example of the form $\omega=A_\mu d\xi^\mu$ (here $p=1$, see relation (16)).

For the form be closed, the differential of the form or its commutator must be equal to zero. The components of the commutator of a form ω can be written as follows:

$$K_{\alpha\beta}=\left(\frac{\partial A_\beta}{\partial \xi^\alpha}-\frac{\partial A_\alpha}{\partial \xi^\beta}\right) \tag{18}$$

here the term connected with the manifold metric form has not yet been taken into account.

The coefficients A_μ of the form ω have been obtained either from the equation of balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes the energetic and force actions have different nature and appear to be inconsistent. The commutator of the form ω made up by the derivatives of such coefficients is nonzero. This means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed.

This is also true for evolutionary form ω^p.

Form ω (as well as form ω^p) is obtained in the accompanying frame of reference connected with the manifold made up by the trajectories of material system elements. This manifold is a nonintegrable deforming manifold. For this reason the form commutator will include one more term connected with the metric form commutator of the manifold. Two terms of the evolutionary form commutator are of different nature and hence cannot compensate one another. Because of this the evolutionary form commutator (and hence, the differential) cannot vanish. That is, the evolutionary form remains to be unclosed.

The nonclosure of evolutionary form ω^p means that this form cannot be a differential (an invariant). For this reason, relation (17), which involves the evolutionary form, appears to be nonidentical (the left-hand side involves a differential, and the right-hand side involves evolutionary form, which is not a differential).

Hence, without a knowledge of particular expression for the form ω^p, one can argue that for actual processes the relation obtained from the equations corresponding to the balance conservation laws proves to be nonidentical.

Relation (17) is just of the same form as the identical relation (9) of exterior differential forms presented in Section 2. However, in the right-hand side of the identical relation (9) stands a closed form, whereas the form in the right-hand side of nonidentical relation (17) is an unclosed one.

Nonidentity of the evolutionary relation obtained from the equations of conservation laws material systems means that the balance conservation law equations are inconsistent. And this points out to the fact that the balance conservation laws are noncommutative [13, 14]. The noncommutativity of conservation laws is just a moving force of evolutionary processes.

3.2.4. Selfvariation of the evolutionary nonidentical relation

The evolutionary nonidentical relation is selfvarying, because, firstly, it is a nonidentical, namely, it contains two terms one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, that is, the variation of any term of the relation in some process leads to a variation of another term; and, in turn, the variation of the latter leads to variation of the former. Since one of the terms is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop.

Selfvariation of the evolutionary relation proceeds by exchange between the evolutionary form coefficients and manifold characteristic. (This is an exchange between physical quantities and space-time characteristic, namely, between the algebraic and geometrical characteristics). Such a process is governed by the commutator of evolutionary form, which, as it has been already noted, in addition to the term made up by derivatives of the evolutionary form coefficients includes the commutator of the manifold metric form, which is nonzero.

3.2.5. Process of obtaining closed exterior forms. Degenerate transformation

A distinction of the evolutionary form from the closed exterior form consists in the fact that the evolutionary differential form is defined on a manifold with unclosed metric forms, and the closed exterior form can be defined only on a manifold with closed metric forms. Hence, it follows that a closed exterior form can be obtained from the evolutionary form only under degenerate transformation (under the transformation that does not conserve the differential), when a transition from the nonintegrable manifold with unclosed metric forms (whose differential is nonzero) to integrable manifold with closed metric forms (for which the differential is zero) takes place.

The degenerate transformation can take place under additional conditions. At describing material systems these conditions can be caused by an availability of any degrees of freedom of material system (such as translational, rotational, oscillatory and others). Such conditions can be realized under selfvariation of a nonidentical relation.

One can see that vanishing the differential of the manifold metric form, that is, realization of closure of the manifold metric form is a condition of degenerate transformation.

The realization of degenerate transformation conditions, that is, to a realization of closure of the manifold metric form is to realization the dual form closure. And this leads to realization pseudostructure and realization of the skew-symmetric form closure on pseudostructure.

Under degenerate transformation the following transition proceeds:

$$d\omega^p \neq 0 \to \text{ (a degenerate transformation) } \to d_\pi \omega^p = 0\,, \;\; d_\pi{}^* \omega^p = 0$$

where the conditions $d_\pi \omega^p = 0$ and $d_\pi{}^* \omega^p = 0$ are the conditions the exterior and dual forms closure.

Realization of the closure conditions for dual form (metric form, which describes a pseudostructure) and realization of the closure conditions for skew-symmetric form points out to that the pseudostructure is realized (a closed dual form $*\,\omega_\pi^p$) and the closed inexact exterior form (ω_π^p) is formatted.

Thus, if the conditions for a degenerate transformation are realized, from an unclosed evolutionary form can obtain a differential form closed on a pseudostructure, that is closed inexact exterior form.

The realization of exterior and dual forms points out to emergence of the differential-geometrical structure (a pseudostructure with a conservative quantity), which is an invariant structure. In the subsection 3.4 the physical meaning of closed form and differential-geometrical structure will be shown.

3.2.6. Conditions of a degenerate transformation.

The Cauchy-Riemann conditions, the characteristic relations, the canonical relations, the Bianchi identities and others are examples of the conditions of degenerate transformations.

Conditions of degenerate transformation are realized at vanishing some functional expressions such as Jacobians, determinants, the Poisson brackets, residues, and others.

The conditions of degenerate transformation specify the integrable surfaces (pseud structures): the characteristic (the determinant of coefficients at the normal derivatives vanishes), the singular points (Jacobian is equal to zero) and so on.

Mathematically, a degenerate transformation is realized as a transition (nonequivalent) from the noninertial frame of reference to the locally-inertial frame of reference. Specially, at describing the material system this is a transition from accompanying (noninertial) frame of reference to pseudostructures. The evolutionary form and nonidentical evolutionary relation are defined in the noninertial frame of reference (on nonintegrable manifold). But the thereby obtained closed exterior form are obtained with respect to the locally-inertial frame of reference (on integrable pseudostructure).

It should be noted that nondegenerate transformations of exterior forms, which conserve the differential, and degenerate transformations of evolutionary forms, which do not conserve the differential, are mutually connected. The degenerate transformations execute the transition from original deformed manifold to pseudostructures and the generation of the differential-geometrical structure. And nondegenerate transformation execute the transition from one differential-geometrical structure to another differential-geometrical structure.

3.2.7. Obtaining an identical relation from a nonidentical one

On pseudostructure π realized, from evolutionary nonidentical relation (17) it follows relation

$$d_\pi \psi = \omega_\pi^p \tag{19}$$

which appears to be an identical relation. Since the form ω_π^p is a closed one, on the pseudostructure this form turns out to be a differential. There are differentials in the left-hand and right-hand sides of this relation. This means that the relation obtained is an identical one. As it will be shown below, such identical relation has a unique physical meaning.

The identity of the relation obtained from evolutionary relation means that the initial equations for material systems (equations of conservation laws) become consistent and integrable on pseudostructures. The pseudostructures made up integrable surfaces (such as characteristics and potential surfaces) on which the quantities of material system sought (such as temperature, pressure and density) become functions of variables only and do not depend on the commutator (and on the way of integrating). These are generalized functions. In this case one can see that the integrable surfaces are obtained from the condition of degenerate transformation of evolutionary relation.

It should be emphasized that under degenerate transformation the evolutionary form differential vanishes only on a pseudostructure. It is an interior differential. The total differential of the evolutionary form is nonzero. The evolutionary form remains unclosed, and for this reason, the original relation, which contains the evolutionary form, remains to be nonidentical one.

It has been shown that from the equations of conservation laws for material media, of which the mathematical physics equations are composed, it is obtained the evolutionary form, which generates closed inexact exterior forms that can describe conservation laws for physical fields. Below it is shown that there exist a connection between the field-theory equations and the mathematical physics equations for material media.

3.3. Connection between the field-theory equations and the mathematical physics equations for material media

As it has been already noted (subsection 2.2), solutions to the field-theory equations are closed inexact exterior forms. It has been shown that such closed inexact exterior forms are obtained from the equations for material media. This points out to that it may exist a connection between the field-theory equations and the equations for material media.

Such a connection is exercised by the nonidentical evolutionary relation.

3.3.1. The identity relations of the mathematical physics equations for material media as an analog the equations of field theories

Functional properties of the field-theory equations

The field-theory equations differ in their functional properties from the mathematical physics equations for material media. The mathematical physics equations are differential equations, its solutions are functions (which describe physical quantities such as a velocity, pressure and density). And the solutions to the field-theory equations are differentials because these equations must describe physical structures, which the closed inexact exterior forms (differentials) are assigned to. Only the equations that have the form of relations may have the solutions which are differentials rather then functions.

One can verify that all equations of existing field theories have the form of relations in differential forms or in the forms of their tensor or differential (i.e. expressed in terms of derivatives) analogs.

At some values of variables from these equations it follows the identical relations, from which the functionals desired are found.

As one can see from the field-theory equations it follows identical relations, which include closed exterior forms or their tensor or differential analogs.

1) The Dirac relations made up of Dirac's bra - and cket - vectors, which connect a closed exterior forms of zero degree;

2) The Poincare invariant, which connects a closed exterior forms of first degree;

3) The relations $d\omega^2 = 0$, $d^*\omega^2 = 0$ for closed exterior forms of second degree obtained from Maxwell equations.

From the Einstein equation it is obtained an identical relation in the case when the covariant derivative of the energy-momentum tensor vanishes.

The identity relations that follow from the evolutionary relation of the equations of mathematical physics are such identity relations

Another specific feature of the field-theory equations consists in the fact that all field-ц цtheory equations are relations for functionals such as a wave function, action functional, Einstein's tensor and so on [12]. (Entropy is such a functional for the fields generated by thermodynamical and gas-dynamical systems.) The relation obtained from the mathematical physics equations for material media is a relation for all these functionals.

The correspondence between the field-theory equations and the relation of mathematical physics equations points to a connection between field theories and the equations for material media. Such correspondence enables one to understand peculiarities and properties of the equations of mathematical physics and existing field theories [15, 16, 17].

Connection of the field-theory equations, which describe physical fields, with the equations for material media points out that it may exist a connection between physical fields and material media.

3.4. Connection of physical fields with material media

The process of realization of closed exterior forms, which describe the conservation laws for physical fields, from evolutionary forms that are obtained from the equations of conservation laws for material media describes the processes, which proceed in material media and are accompanied by origination of physical structures.

3.4.1. Mechanism of evolutionary processes in material media. Origination of the physical structures

Physical meaning of evolutionary relation

The evolutionary relation includes the differential $d\psi$ that specifies the state of material media. Since the relation is nonidentical (this is due to noncommutativity of the conservation laws), one cannot obtain the differential from this relation. This points to the fact of absence of the state function and the nonequilibrium state of the media.

The nonequilibrium state of the media means that there is an internal force in the material media. It is evident that the internal force originates at the expense of some quantity described by the evolutionary form commutator. (If the balance conservation laws be commutative, the evolutionary form commutator be zero, and this would point to the equilibrium state, i.e. the absence of internal forces.) Everything that gives a contribution to the evolutionary form commutator leads to emergence of internal force. [In particular, this can arise at the expense of energetic or force action to the media [13]. In this case, the internal force arises due to the fact that energetic and force actions cannot directly convert into measurable physical quantities (energy and impulse) of the system itself and is accumulated in the form of some unmeasurable quantity, which acts as internal force.

Evolutionary relation also describes a variation of nonequilibrium state. Selfvariation of the nonidentical evolutionary relation points to the fact that the state of medium turns out to be selfvarying. It is evident that this selfvariation proceeds under the action of internal force whose quantity is described by the commutator of unclosed evolutionary form ω^p. (It proceeds a deformation of the manifold made up by the trajectories of the material medium elements and a change of medium characteristics.) In this process the state of medium changes but remains nonequilibrium because the internal forces do not vanish due to that the evolutionary form commutator remains to be nonzero.

Here it should be noted that in a real physical process the internal forces can be increased (due to selfvariation of the nonequilibrium state of material medium). This can lead to development of instability in medium. {For example, this was pointed out in the works by Prigogine [18]. In his works «the excess entropy" is an analog to the commutator of nonintegrable form for thermodynamic system. “Production of excess entropy" leads to development of instability.

The transition from nonidentical relation to identical one points to the transition of material medium from nonequilibrium state to locally-equilibrium state. From identical relation one can obtain the differential of state. This points to availability of the state function and the transition of material medium to locally equilibrium state (such a transition is a local one since the identical relation is realized only on a certain pseudostructure).

The transition from nonequilibrium state to locally-equilibrium one means that the material medium has been locally got rid of internal force. This follows from the evolutionary relation. As it has been shown, the transition from nonidentical relation to identical one is realized under degenerate transformation when it is proceeded a transition from evolutionary form with the commutator (which describes internal forces) being nonzero to the closed exterior form with vanishing commutator, and this points out to an absence of internal force.

Emergence of observable formations in material medium. Origination of the physical structures

Realization of the conditions of degenerate transformation, which describes a transition of material medium from nonequilibrium state to locally-equilibrium one, are conditioned by the degrees of freedom of material medium. That is, the transition of material medium from noneqilibrium state to equilibrium one proceeds under realization of any degree of freedom. The presence of the degree of freedom allows the inconsistent quantities to redistribute and transform to measurable quantities of the medium itself. In particular, because of noncommutatibity of conservation laws for energy and linear momentum the energetic and force actions (onto local domain of material medium) cannot directly convert into measurable quantities (energy and impulse) of material medium itself. This is possible only at presence of any degree of freedom, this makes it possible a redistribution and obtaining consistent energy and impulse.

The measurable quantities of material medium realized reveal in material medium as the spontaneous emergence of observable measurable formations. Waves, vortices, fluctuations, turbulent pulsations and so on are examples of such formations [19].

One can see that the transition of material medium from nonequilibrium state the locally-equilibrium one is accompanied by the emergence of observable formations in material medium.

On the other hand, as it has been shown, the transition from nonidentical relation to identical one, which describes the transition of material medium from nonequilibrium state the locally-equilibrium one, occurs at realization closed inexact exterior and dual forms. This points out to a realization of a differential-geometrical structure, namely, a pseudostructure with conservative quantity, which corresponds to the conservation law. The physical structures that form physical fields are just such differential-geometric structures, which correspond to the conservation laws for physical fields (conservative quantities or objects). (Massless particles, structures made up by eikonal surfaces and wave fronts, and so on are examples of physical structures.)

Thus can see, that the transition of material system from nonequilibrium state the locally-equilibrium one is accompanied by origination of physical structures and the emergence of observable formations in material media.

It shows that the physical structures, which constitute physical fields, are generated by material media.

Physical structures and observed formations are a manifestation of the same phenomenon. (Light is an example of manifestation of such a duality, namely, as a massless particle (photon) and as a wave.) However, physical structures and observed formations are not identical objects. Whereas the wave is an observable formation, to the physical structure it is assigned the eikonal (that is, the moving wave element made up a physical structure).

The characteristics of physical structure and formations are determined by the evolutionary form, the evolutionary form commutator, and additional conditions, which are specified by the degenerate transformation. The connection of physical structures with skew-symmetric differential forms allows to introduce a classification of these structures in dependence on parameters that specify skew-symmetric differential forms. Realization of pseudostructures (connected with origination of physical structure and fulfilling the conservation law) is a manifestation of the mechanism of formatting pseudo metric and metric manifolds [20].

Duality of physical structures and observed formations is described by identical relation (19). The left-hand side of this relation includes the differential, which specifies material system and whose availability points to the locally-equilibrium state of material media and an emergence of measurable formations. And the right-hand side includes a closed inexact form, which is a characteristic of physical structures.

It is evident that the identical relation obtained from nonidentical evolutionary relation has a unique meaning. This relation discloses a connection between physical fields and material media.

It should emphasize once more that the connection of physical fields with material systems is realized by the conservation laws. The closed inexact exterior forms, which describe the conservation laws for physical fields, are obtained from the evolutionary forms, which enter into nonidentical relation derived from the equations of noncommutative conservation laws for material systems. It is just the noncommutativity of balance conservation laws plays a governing role in evolutionary processes, which proceed in material media and lead to generating physical fields.

To obtain physical structures, which produce given physical field, one has to consider the material media assigned to this field and the relevant evolutionary relation. In particular, to obtain the thermodynamic structures it is necessary to consider the evolutionary relation for thermodynamic media, to obtain gas-dynamic structures – the relation for gas-dynamic media, for electromagnetic field – the relation obtained from the equations for charged particles, for gravitational field – the relation for cosmologic systems.

4. Conclusion

In conclusion it should be emphasized once more the unique role of the skew-symmetric differential forms for field theory. The closed exterior forms describe the properties that are common for all field theories, and this discloses an internal connection of existing field theories and may serve as an approach to constructing a unified field theory. And the evolutionary forms (whose existence has been established by the author), discloses the basis on which field theories are build up (the connection between field theories equations and the mathematical physics equations for material media).

The theory of skew-symmetric differential forms, which discloses the internal connection between existing field theories and their basis, cam help in solving the problems of existing field theories and building the general field theory.

The process of realization of closed exterior forms describes the mechanism of evolutionary processes, which proceeds in material media and are accompanied by an origination of physical structures, that made up physical fields. This points out to the fact that material media generate physical fields.

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