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Review Article

Two-Mode Sub harmonic Generator Coupled to Thermal Reservoir

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Abstract: In this article, our investigation to study squeezing and statistical properties of the light by a two mode sub harmonic generator coupled to thermal reservoir via a single port-mirror. The equation of motion answers are then used to calculate the mean photon number, photon number variance, and quadrature variance for two mode cavity light. However, we have found that the degree of squeezing is indeed affected by the present of thermal light. The mean photon number of the system under consideration increases with increasing $\bar{n}$.

Keywords: Master Equation, Mean Photon, Density Operator and Q-Function, Quadrature Fluctuation, Quadrature Squeezing

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1. Introduction

Over the years, a considerable attention has been paid to squeeze states of light. Squeezed state was first theoretically predicted [1-8] and subsequently experiment observed [4-20]. In a squeezed state the quantum noise in one quadrature is below the coherent-state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadrature's satisfying the uncertainty relation [18, 19]. The interaction of coherent light with non-linear crystal leads to the generation of squeezed light. With the aid of the pertinent Hamiltonian, we first determine the master equation and c-number Langevin equation for the two mode sub harmonic generator coupled to thermal reservoir. Employing the solution of the c-number Langevin equations, we obtain the Q function. In this process a pump photon of frequency 2ω is down converted into a pair of signal photons each of frequency ω. On the other hand, two-mode sub harmonic generator, consisting of a non-linear crystal pumped by coherent light is placed in a cavity coupled to a vacuum reservoir, is a prototype source of a two mode squeezed light [8-27]. In this system a photon of frequency ω_c is down converted in to a pair of highly correlated signal-idler photons having each of frequency ω_a and ω_b respectively [1-27]. It has been established that the signal mode has a maximum of 50 squeezing below the coherent state level [1-7]. Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Squeezing is one of the interesting non classical features of light that has been attracting attention and studied by many authors. In squeezed light the noise in one quadrature is below the vacuum or coherent state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadrature's satisfying the uncertainty relation. Squeezed light has potential applications in low-noise communications and precision measurements [13, 14]. A sub harmonic generator has been considered as an important source of squeezed light. It is one of the most interesting and well characterized optical devices in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency the device is called a one mode sub harmonic generator, otherwise it is called a two mode sub harmonic generator. The quantum dynamics of a one mode sub harmonic generator coupled to two uncorrelated squeezed vacuum reservoirs has been analyzed employing the Q function obtained by solving the Fokker-Planck equation using the propagator method [7, 15]. The variance of the quadrature operators and the photon number distribution for the signal-idler modes produced by a two mode sub harmonic generator coupled to a two-mode squeezed vacuum reservoir have also been studied applying the pertinent Langevin equations [3]. On the other hand, obtaining stochastic differential equations, associated with the normally ordering, for the cavity mode variables appears to involve a relatively less mathematical task. In view of this, the main objective of this study, employing c-number langevin equations, the squeezing and statistical properties of the light produced by a two mode sub harmonic generator coupled to a two mode thermal reservoir via a single port-mirror to be analyzed. We first obtain stochastic differential equations for the cavity mode variables by applying the pertinent master equation. In addition, with the aid of the Q function, we calculate the mean photon number, the variance of the photon number, the quadrature variance, the quadrature squeezing, and the photon number distribution.

2. The Q Function

A. The master equation

We first obtain the master equation, for the signal-idler modes produced by the two-mode sub harmonic generator coupled to thermal reservoir (as shown in Figure 1).

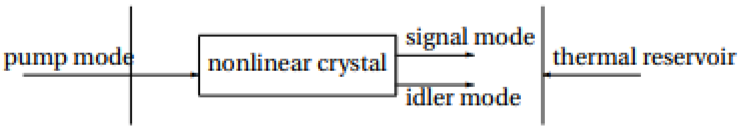

Figure 1. Two-mode sub harmonic generator coupled to thermal reservoir.

Then using the master equation, we obtain c-number Langavin equations, associated with normal ordering.

The process of two-mode sub harmonic generation is described by the Hamiltonian [1-8].

$$\hat{H}_s = i\mu(\hat{c}^+ - \hat{c}) + i\lambda(\hat{a}\hat{b}\hat{c}^+ - \hat{a}^+\hat{b}^+\hat{c}) \tag{1}$$

in which, $\hat{a}^+$, $\hat{b}^+$ and $\hat{c}^+$ are creation operators for the signal, idler, and pump mode, respectively. λ is the coupling constant, and μ is proportional to the amplitude of the coherent light driving the pump mode. With the pump mode represented by a real and constant c-number γ, the process of two-mode sub harmonic generation can be described by the Hamiltonian [9-15].

$$\hat{H}_s = i\varepsilon(\hat{a}\hat{b} - \hat{a}^+\hat{b}^+) \tag{2}$$

where, $\varepsilon = \lambda\gamma$.

On other hand, the master equation for a cavity mode coupled to a reservoir can be written as [2].

$$\frac{d\hat{\rho}}{dt} = i[\hat{H}sR, \hat{\rho}(t)] - h\langle \hat{H}^2SR\rangle \hat{\rho}(t) - h\rho(t)\langle \hat{H}^2 SR\rangle + 2hTr(\hat{H}SR\hat{\rho}(t)\hat{H}SR \tag{3}$$

The interaction Hamiltonian for a two mode cavity light to a reservoir is given by

$$\hat{H}_{SR} = i\lambda(\hat{a}^{+}\hat{a}_{in} - \hat{a}^{+}{}_{in}\hat{a} + \hat{b}^{+}\hat{b}_{in} - \hat{b}^{+}{}_{in}\hat{b} \tag{4}$$

Taking the square of Equation. 4 and then the expectation value of it, we observe that

$$\hat{H}^{2}{}_{SR}\rangle R = \langle (i\lambda(\hat{a}^{+}\hat{a}_{in} - \hat{a}^{+}{}_{in}\hat{a} + \hat{b}^{+}\hat{b}_{in} - b^{+}{}_{in}\hat{b}))^{2})\rangle \tag{5}$$

Applying the fact that the cavity mode operators and operators of thermal reservior are commute to each other.
Employing the density operator for a chaotic light given as

$$\hat{\boldsymbol{\rho}}_{\mathbf{R}} = \sum_{\mathbf{n=0}}^{\infty} \frac{\bar{\boldsymbol{n}}^{\mathbf{n}}}{(\mathbf{1}+\bar{\boldsymbol{n}})^{\mathbf{n+1}}} |\mathbf{n}\rangle\langle\mathbf{n}| \tag{6}$$

Where, n = 0; 1; 2; 3; 4….. Is number of integers and $\bar{\boldsymbol{n}}$ is the mean photon number of two-mode cavity light coupled to a reservoir.
One can easily write

$$\langle \hat{a}^{2}{}_{in}R\rangle = TrR(\hat{R}\,\hat{a}^{2}{}_{in}). \tag{7}$$

Thus introducing Equation. (6) in (7), we get

$$\langle\, \hat{a}^{2}{}_{in}\,\rangle = \sum_{n=0}^{\infty} \frac{\bar{n}^{n}}{(1+\bar{n})^{\,n+1}} TrR(|n\rangle\langle n|\hat{a}^{2}{}_{in}) = 0 \tag{8}$$

In which

$$\langle n|n-2\,\rangle = 0. \tag{9}$$

One can also check that

$$\langle \hat{a}^{2}{}_{in}\rangle = \langle\, \hat{b}^{2}{}_{in}\rangle = \langle\, \hat{a}^{+2}{}_{in}\rangle = \langle \hat{b}^{+2}{}_{in}\rangle = 0, \tag{10}$$

Because, the expectation value of an operator with its self is zero. In addition, applying the commutation relation

$$[\hat{a}_{in}\,, \hat{a}^{+}{}_{in}] = 1 \tag{11}$$

We then note that

$$\langle \hat{a}_{in}\hat{a}^{+}{}_{in}\rangle = \bar{n} + 1 \tag{12}$$

With

$$\langle \hat{a}^{+}{}_{in}\,\hat{a}_{in}\rangle = \bar{n}. \tag{13}$$

In which $\bar{\boldsymbol{n}}$ is the mean photon number of the thermal reservoir. Hence upon substituting Equations. (5), (7), (8), (10), and (11) into (4), we get

$$hTrR\langle \hat{H}^{2}{}_{SR}\rangle\hat{\rho} = h\lambda^{2}TrR((\bar{n}+1)(\hat{a}^{+}\hat{a}\,\hat{\rho} + \hat{b}^{+}\hat{a}\hat{\rho}) + \bar{n}(\hat{a}\hat{a}^{+}\hat{\rho} + \hat{b}\hat{b}^{+}\hat{\rho})) \tag{14}$$

Following the same manner, we obtain

$$h\hat{\rho}TrR\langle \hat{H}^{2}{}_{SR}\rangle = h\lambda^{2}TrR((\bar{n}+1)(\hat{\rho}\hat{a}^{+}\hat{a} + \hat{\rho}\,\hat{b}^{+}\hat{b}) + \bar{n}(\hat{\rho}\hat{a}\hat{a}^{+} + \hat{\rho}\hat{b}\hat{b}^{+})) \tag{15}$$

And

$$2hTrR(\hat{H}_{SR}\hat{\rho}(t)\hat{\rho}\hat{H}_{SR}) = 2\lambda^{2}h(\bar{n}(\hat{a}^{+}\hat{\rho}\hat{a} + \hat{b}^{+}\hat{\rho}\hat{b}) + (\bar{n}+1)(\hat{a}\hat{\rho}\hat{a}^{+} + \hat{b}\hat{\rho}\hat{\mathbf{b}}^{+}) \tag{16}$$

Thus employing Equations. (4), (14), (15) and (16), we readily obtain the master equation for a cavity mode coupled to thermal reservoir as in the form

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] + \frac{K}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}) + \frac{K}{2}\bar{n}(2\hat{a}^+\hat{\rho}\hat{a} - \hat{a}\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}) +$$

$$\frac{K}{2}(\bar{n}+1)(2\hat{b}\hat{\rho}\hat{b}^+ - \hat{b}^+\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^+\hat{b}) + \frac{K}{2}\bar{n}\ (2\hat{b}^+\hat{\rho}\hat{b} - \hat{b}\hat{b}^+\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^+). \quad (17)$$

where K = $2\lambda^2$ h is the cavity damping constant and assuming that the cavity damping constant is taken to be the same, i.e. $\kappa a = \kappa b = \kappa$ and $\bar{\boldsymbol{n}}a = \bar{\boldsymbol{n}}b = \bar{\boldsymbol{n}}$. With the aid of Equation. (2), the reduced density operator can be put in the form

$$\frac{d\hat{\rho}(t)}{dt} = \varepsilon(\hat{a}\hat{b}\hat{\rho} - \hat{a}^+\hat{b}^+\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^+\hat{b}^+) + \frac{K}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a})$$

$$+\frac{K}{2}(\bar{n}+1)(2\hat{b}\hat{\rho}\hat{b}^+ - \hat{b}^+\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^+\hat{b}) - \frac{K}{2}\bar{n}(2\hat{a}^+\hat{\rho}\hat{a} - \hat{a}\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^+)$$

$$+\frac{K}{2}\bar{n}(2\hat{b}^+\hat{\rho}\hat{b} - \hat{b}\hat{b}^+\hat{\rho}) - \hat{\rho}\hat{b}\hat{b}^+) \quad (18)$$

This is the master equation for the two-mode sub harmonic generator coupled to thermal reservoir.

B. c-number Langavin equation

We then seek to obtain the operator dynamics applying the master equation. To this end, employing the relation

$$\frac{d}{dt}\langle \hat{A} \rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}) \quad (19)$$

Where, $\hat{\boldsymbol{A}}$ is linear gain coefficient and the commutation relation,

$$[\hat{a}, \hat{a}^+] = [\hat{b}, \hat{b}^+] = 1. \quad (20)$$

Along with Equation. (18), we can readily obtain

$$\frac{d}{dt}\langle \hat{a}(t) \rangle = -\varepsilon\langle \hat{b}^+(t)\rangle - \frac{K}{2}\langle \hat{a}(t)\rangle \quad (21)$$

$$\frac{d}{dt}\langle \hat{b}(t)\rangle = -\varepsilon\langle \hat{a}^+(t)\rangle - \frac{K}{2}\langle \hat{b}(t)\rangle \quad (22)$$

$$\frac{d}{dt}\langle \hat{a}^2(t)\rangle = -2\varepsilon\langle \hat{a}(t)\hat{b}^+(t)\rangle - K\langle \hat{a}^2(t)\rangle \quad (23)$$

$$\frac{d}{dt}\langle \hat{b}^2(t)\rangle = -2\varepsilon\langle \hat{b}(t)\hat{a}^+(t)\rangle - K\langle \hat{b}^2\rangle \quad (24)$$

$$\frac{d}{dt}\langle \hat{a}^+(t)\hat{a}(t)\rangle = -\varepsilon\langle \hat{a}(t)\hat{b}(t)\rangle - \varepsilon\langle \hat{a}^+(t)\hat{b}^+(t)\rangle - K\langle \hat{a}^+(t)\hat{a}(t) + K\bar{n}\rangle \quad (25)$$

$$\frac{d}{dt}\langle \hat{b}^+(t)\hat{b}(t)\rangle = -\varepsilon\langle \hat{a}(t)\hat{b}(t)\rangle - \varepsilon\langle \hat{a}^+(t)\hat{b}^+(t)\rangle - K\langle \hat{b}^+(t)\hat{b}(t)\rangle + K\bar{n} \quad (26)$$

$$\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t)\rangle = -\varepsilon\langle \hat{a}^+(t)\hat{a}(t)\rangle - \varepsilon\langle \hat{b}^+(t)\hat{b}(t)\rangle - \varepsilon - K\langle \hat{a}(t)\hat{b}(t)\rangle. \quad (27)$$

$$\frac{d}{dt}\langle \hat{a}^{+}(t)\hat{b}(t)\rangle = -\varepsilon\langle \hat{b}^{2}(t)\rangle - \varepsilon\langle \hat{a}^{+2}(t)\rangle - K\langle \hat{a}^{+}(t)\hat{b}(t)\rangle. \tag{28}$$

$$\frac{d}{dt}\left\langle \hat{b}^{+}(t)\hat{a}(t)\right\rangle = -\varepsilon\langle \hat{a}^{2}(t)\rangle - \varepsilon\left\langle \hat{b}^{+2}(t)\right\rangle - K\langle \hat{b}^{+}(t)\hat{a}(t)\rangle \tag{29}$$

Then c-number function corresponding to Equations. (21-29) is

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\varepsilon\langle \beta^{*}(t) > -\frac{K}{2}\langle \alpha(t)\rangle \tag{30}$$

$$\frac{d}{dt}\langle \beta(t)\rangle = -\varepsilon\langle \alpha^{*}(t) > -\frac{K}{2}\langle \beta(t)\rangle \tag{31}$$

$$\frac{d}{dt}\langle \alpha^{2}(t)\rangle = -2\varepsilon\langle \alpha(t)\beta^{*}(t)\rangle - K\langle \alpha^{2}(t)\rangle \tag{32}$$

$$\frac{d}{dt}\langle \beta^{2}(t)\rangle = -2\varepsilon\langle\beta(t)\alpha^{*}(t)\rangle - K\langle \beta^{2}(t)\rangle \tag{33}$$

$$\frac{d}{dt}\langle \alpha^{*}(t)\alpha(t)\rangle = -\varepsilon\langle \alpha(t)\beta(t)\rangle - \varepsilon\langle \alpha^{*}(t)\beta^{*}(t)\rangle - K\langle \alpha^{*}(t)\alpha(t)\rangle + K\bar{n}. \tag{34}$$

$$\frac{d}{dt}\langle \beta^{*}(t)\beta(t)\rangle = -\varepsilon\langle \alpha(t)\beta(t)\rangle - \varepsilon\langle \alpha^{*}(t)\beta^{*}(t)\rangle - K\langle \beta^{*}(t)\beta(t)\rangle + K\bar{n}. \tag{35}$$

$$\frac{d}{dt}\langle \alpha(t)\beta(t)\rangle = -\varepsilon\langle \alpha^{*}(t)\alpha(t)\rangle - \varepsilon\langle \beta^{*}(t)\beta(t)\rangle - \varepsilon - K\langle \alpha(t)\beta(t)\rangle. \tag{36}$$

$$\frac{d}{dt}\langle\alpha^{*}(t)\beta(t)\rangle = -\varepsilon\langle \beta^{2}(t)\rangle - \varepsilon\langle \alpha^{*2}(t)\rangle - K\langle \alpha^{*}(t)\beta(t)\rangle. \tag{37}$$

$$\frac{d}{dt}\langle \beta^{*}(t)\alpha(t)\rangle = -\varepsilon\langle \alpha^{2}(t)\rangle - \varepsilon\langle \beta^{*2}(t)\rangle - K\langle \beta^{*}(t)\alpha(t)\rangle. \tag{38}$$

On the basis of Equations. (30) and (31), we can write

$$\frac{d}{dt}\alpha(t) = -\varepsilon\beta^{*}(t) - \frac{K}{2}\alpha(t) + f_{\alpha}(t) \tag{39}$$

And

$$\frac{d}{dt}\beta(t) = -\varepsilon a^{*}(t) - \frac{K}{2}\beta(t) + f_{\beta}(t). \tag{40}$$

Where $f\alpha$ (t) and $f\beta$ (t) are the noise forces whose correlation properties remain to be determined. Taking the expectation value of Equations. (39) and (40), we see that

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\varepsilon\beta^{*}(t) - \frac{K}{2}\langle \alpha(t)\rangle + f\alpha(t) \tag{41}$$

And

$$\frac{d}{dt}\langle \beta(t)\rangle = -\varepsilon a^{*}(t) - \frac{K}{2}\langle \beta(t)\rangle + f\beta(t). \tag{42}$$

Comparing Equations. (30) and (41) as well as Equations. (31) and (42), we observe that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0 . \tag{43}$$

To determine the correlation properties of the noise forces, we introduce the mathematical relation

$$\frac{d}{dt}\langle \gamma(t)\zeta(t)\rangle = \langle \left(\frac{d}{dt}\gamma(t)\zeta(t)\right)\rangle + \langle \gamma(t)\left(\frac{d}{d}\zeta(t)\right)\rangle. \tag{44}$$

Applying this relation, we can write Equation. (30) as

$$\frac{d}{dt}\langle \alpha(t)\alpha(t)\rangle = \langle\left(\frac{d}{dt}\alpha(t)\right)\alpha(t)\rangle + \langle \alpha(t)\left(\frac{d}{dt}\alpha(t)\right)\rangle . \tag{45}$$

Inspection of Equations. (30) and (45) indicate that

$$\langle f_\alpha(t)\alpha(t)\rangle + \langle \alpha(t)f_\alpha(t)\rangle = 0. \tag{46}$$

The formal solution of Equation. (39) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{K}{2}t} - \int_0^t e^{-\frac{K}{2}(t-t')}[-\varepsilon\beta^*(t) - f_\alpha(t')]dt' . \tag{47}$$

But a noise force at a later time does not affect c-number variable in earlier time, hence we observe that

$$\langle \alpha(0)f_\alpha(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha(t)\rangle . \tag{48}$$

Thus Equation. (41) leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = \int_0^t e^{-\frac{K}{2}(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle\, dt'. \tag{49}$$

Also multiplying Equation. (41) by fα (t) from the left at both sides and taking the expectation value, we have

$$\langle f_\alpha(t)\alpha(t)\rangle = \langle f_\alpha(t)\alpha(0)\rangle e^{-\frac{K}{2}(t)} - \int_0^t e^{-\frac{K}{2}(t-t')}[-\varepsilon(\langle f_\alpha(t)\beta^*(t')\rangle) - \langle f_\alpha(t)f_\alpha(t')\rangle]\, dt'. \tag{50}$$

Following the same procedure, Equation. (50) becomes

$$\langle f_\alpha(t)\alpha(t)\rangle = \int_0^t e^{-\frac{K}{2}(t-t')}\langle f_\alpha(t)f_\alpha(t')\rangle dt'. \tag{51}$$

Assuming

$$\langle f_\alpha(t)f_\alpha(t')\rangle = \langle f_\alpha(t')(t)\rangle . \tag{52}$$

And using Equations. (51) and (50), Equation. (49) yields

$$\langle \alpha(t)f_\alpha(t)\rangle + \langle f_\alpha(t)\alpha(t)\rangle = 2\int_0^t e^{-\frac{K}{2}(t-t')\langle f_\alpha(t)f_\alpha(t')\rangle dt'}. \tag{53}$$

Now applying the relation [2]

$$\int_0^t e^{\frac{K}{2}(t-t')\langle f_\alpha(t)g_\alpha(t')\rangle}dt' = d . \tag{54}$$

We assert that

$$\langle\ f_\alpha(t)g_\alpha(t')\ \rangle = 2d\delta(t-t')\,. \tag{55}$$

Thus on account of Equation. (52), we see that

$$\langle\ f_\alpha(t)f_\alpha(t')\rangle\ =\ \langle\ f_\alpha(t')f_\alpha(t)\rangle\ = 0\,. \tag{56}$$

Following the same procedure, we find

$$\langle\ f_\beta(t)f_\beta(t')\rangle\ =\ \langle\ f_\beta(t')f_\beta(t)\ \rangle = 0, \tag{57}$$

$$\langle\ f^*{}_\alpha(t')f_\beta(t)\ \rangle\ =\ \langle\ \ f^*{}_\alpha(t)f_\beta(t')\ \rangle = 0\,. \tag{58}$$

Furthermore, it can be easily verified employing Equation. (34) that

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle\ = -2\varepsilon\,\langle\beta(t)\alpha(t)\rangle - K\langle\alpha^*(t)\alpha(t)\rangle + \langle\ f^*{}_\alpha(t)\alpha(t)\rangle + \langle\alpha^*(t)f^*{}_\alpha(t)\rangle\,. \tag{59}$$

Now comparison of Equations s. (36) and (59), we observe that

$$\langle f^*{}_\alpha(t)\alpha(t)\rangle + \langle\alpha^*(t)f^*{}_\alpha(t)\ \rangle = K\bar{n}\ . \tag{60}$$

Multiplying Equation. (47) by $f\alpha^*(t)$ from the left at both side and taking the expectation value, we have

$$\langle f^*{}_\alpha(t)\alpha(t)\rangle\ = \langle\ f^*{}_\alpha(t)\alpha(0)\rangle e^{-\frac{K}{2}t} - \int_0^t e^{-\frac{K}{2}(t-t')}\ [-\varepsilon\langle\ f^*{}_\alpha(t)\beta^*(t)\rangle -\ \langle f^*{}_\alpha(t)f_\alpha(t')\rangle]dt'. \tag{61}$$

Since a noise force at later time does not affect c-number variable in earlier time, so that Then Equation. (61) becomes

$$\langle f^*{}_\alpha(t)\alpha(t)\rangle\ = e^{-\frac{K}{2}(t-t')}\langle f^*{}_\alpha(t)f_\alpha(t')\rangle dt'. \tag{62}$$

Introducing the complex conjugate of Equation. (62) and multiplying it by $f\alpha\ (t)$ and taking the expectation value, we have

$$\langle\alpha^*(t)f_\alpha(t)\rangle\ = e^{-\frac{K}{2}(t-t')}\langle\ f^*{}_\alpha(t')f_\alpha(t)\rangle dt', \tag{63}$$

Assuming

$$\langle f^*{}_\alpha(t)f_\alpha(t')\rangle\ = \langle\ f_\alpha{}^*(t')f_\alpha(t)\rangle dt'. \tag{64}$$

Adding Equations. (61) and (63), we get

$$\langle f^*{}_\alpha(t)\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle\ = 2e^{-\frac{K}{2}(t-t')}\langle f^*{}_\alpha(t)f_\alpha(t')\rangle dt'. \tag{65}$$

In view of Equations. (63) and (64) leads to

$$e^{-\frac{K}{2}(t-t')}\ \langle f^*{}_\alpha(t)f_\alpha(')\rangle\ = \frac{K}{2}\bar{n}. \tag{66}$$

Thus on account of Equations. (56) and (66), we assert that

$$\langle f^*{}_\alpha(t)f_\alpha(t')\ \rangle = \langle f^*{}_\alpha(t')f_\alpha(t)\rangle\ = K\bar{n}\delta(t-t')\,, \tag{67}$$

It can also be verified following a similar procedure that

$$\langle f^*{}_\beta(t)f_\beta(t')\rangle\ = \langle f^*{}_\beta(t')f_\beta(t)\rangle\ = K\bar{n}\delta(t-t')\,. \tag{68}$$

Moreover, with the aid of Equation. (36), we see that

$$\frac{d}{dt}\langle \alpha(t)\beta(t) \rangle = -\varepsilon \langle\beta^*(t)\beta(t)\rangle - \varepsilon\langle \alpha(t)\alpha^*(t)\rangle - K\langle \alpha(t)\beta(t)\rangle + \langle\alpha(t)f_\beta(t)\rangle + \langle f_\alpha(t)\beta(t)\rangle \quad (69)$$

Upon comparing Equations. (34) and (69), we notice that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle f_\alpha(t)\beta(t)\rangle = -\varepsilon\,, \quad (70)$$

The formal solution of Equation. (38) can be written as follows

$$\beta(t) = \beta(0)e^{-\frac{K}{2}(t)} - \int_0^t e^{-\frac{K}{2}(t-t')}\left[-\varepsilon\alpha^*(t') - f_\beta(t')\right]dt', \quad (71)$$

Then multiplying Equations. (48) and (71) by fβ (t) and fα (t) from the right and the left hand side at both side, respectively and taking their expectation value, we get

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{\frac{-K}{2}(t)} - \int_0^t e^{-\frac{K}{2}(t-t')}\left[-\varepsilon\left(\langle\beta^*(t')f_\beta(t)\rangle\right) - \langle f_\alpha(t')f_\beta(t)\rangle\right]dt'\,. \quad (72)$$

It can also be established in a similar manner that

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\alpha(t')f_\beta(t)\rangle = -\varepsilon\delta(t-t'). \quad (73)$$

In order to obtain the solution of Equations. (37) and (38), we introduce a new variable define by

$$\Gamma_\pm(t) = \alpha(t) \pm \beta^*(t)\,, \quad (74)$$

Applying Equation. (37) along with the complex conjugate of Equation. (38), we readily obtain

$$\frac{d}{dt}\Gamma\pm(t) = -\frac{1}{2}\zeta\pm + f_\alpha(t) + f_\beta(t)\,, \quad (75)$$

In which

$$\zeta\pm(t) = K\pm\varepsilon\,. \quad (76)$$

According to Equations. (75) and (76), the equation of evolution of $\boldsymbol{\Gamma}-$ does not have a well behaved solution for K < 2ε. We then identify K = ε as a threshold condition. For 2 ε < K, the solution of Equation. (75) Can be written as

$$\Gamma_\pm = \Gamma_\pm(0)e^{-\frac{\zeta\pm(t)}{2}(t)} + \int_0^t e^{-\frac{\zeta\pm(t)}{2}(t-t')}(f_\alpha(t')\beta^*(t))\,dt'\,. \quad (77)$$

It then follows that

$$\alpha(t) = A_+(t)\alpha(0) + A_-(t)\beta^*(t) + \beta_+(t) + \beta_-(t), \quad (78)$$

$$\beta(t) = A_+(t)\beta(0) + A_-(t)\alpha^*(0) + B^*{}_+(t)B^*{}_-(t)\,. \quad (79)$$

Where

$$A_\pm(t) = \frac{1}{2}\left(e^{-\frac{\zeta\pm}{2}(t)} \pm e^{-\frac{\zeta\pm}{2}(t)}\right), \quad (80)$$

And

$$B_\pm(t) = \frac{1}{2}\int_0^t e^{-\frac{\zeta\pm}{2}(t-t')}\left(f_\alpha(t) \pm f_\beta{}^*(t')\right)dt'\,. \quad (81)$$

C. The Q function

The Q function for a two-mode cavity light can be defined as [2].

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2zd^2\eta\varphi_a\,(z,\eta\,,t)\exp[z^*\alpha \;+\; \eta * \beta - z\alpha^* - \;\eta\beta^*], \tag{82}$$

Where the anti-normally ordered characteristic function φ_a (z, η, t) for the two mode cavity light is given by [2]

$$\varphi_a(Z,\eta,t) = \mathrm{Tr}\left(\rho(0)e^{-z^*a(t)}e^{-\eta^*b(t)}e^{za^+(t)}e^{\eta b^+(t)}\right). \tag{83}$$

Now we see that

$$\varphi_a(z,\eta,t) = \exp[z^*z - \eta^*\eta]\,\mathrm{Tr}\left(\rho(0)\exp\left[z\hat{a}^+(t) + \eta\hat{b}^+(t)\right) - x^*\hat{a}(t) - \eta^*\hat{b}(t)\right]. \tag{84}$$

It is possible to express Equation. (84) in terms of c-number variable associated with the normal ordering as

$$\varphi_a\,(z,\eta,t) = \exp[-z^*z - \;\eta^*\eta]\,\langle\exp[z\alpha^*(t) + \eta\beta^*(t) - z^*\alpha(t) - \eta^*\beta(t)]\,\rangle. \tag{85}$$

Since α (t) and β (t) are Gaussian variables with zero mean, then, the expectation values of the c-number variables appeared in Equation. (84) Can be determining by using Equations. (78) And (79), Employing Equation. (78) and taking their complex conjugate, we get

$$\langle\alpha^{*2}(t)\rangle \;= A^{*2}{}_+(t)\,\langle\alpha^{*2}(0)\rangle + A^*{}_+(t)A^*{}_-(t)\,\langle\alpha^*(0)\beta(0)\rangle + A^*{}_+(t)\,\langle B^*{}_+(t)\alpha^*(0)\rangle + A^*{}_+(t)\,\langle B^*{}_-(t)\alpha^*(0)\rangle + A^*{}_-(t)A^*{}_+(t)\,\langle\beta(0)\alpha^*(0)\rangle + A^{*2}{}_-(t)\,\langle\beta^2(0)\rangle + A^*{}_-(t)\,\langle B^*{}_+(t)\beta(0)\rangle + A^*{}_-(t)\,\langle B^*{}_-(t)\beta(0)\rangle + A^*{}_+(t)\,\langle B^*{}_+(t)\alpha^*(0)\rangle + A^*{}_-(t)\,\langle B^*{}_+(t)\beta(0)\rangle + \langle B^*{}_+(t)B^*{}_+(t)\rangle + B^*{}_+(t)B^*{}_-(t) + A^*{}_+(t)\,\langle B^*{}_-(t)\alpha^*(0)\rangle + A^*{}_-(t)\,\langle B^*{}_-(t)\beta(0)\rangle + \langle B^*{}_-(t)B^*{}_+(t)\rangle + \langle B^*{}_-(t)B^*{}_-(t)\rangle. \tag{86}$$

We assume that the cavity mode is initially in a two-mode vacuum state as well as the noise force at early time does not affect the c-number variable at later time, Equation. (86) Reduced to

$$\langle\alpha^{*2}(t)\rangle \;=\; \langle B^*{}_+(t)B^*{}_+(t)\rangle + \langle\, B^*{}_+(t)B^*{}_-(t)\rangle + \langle\, B^*{}_-(t)B^*{}_+(t)\rangle \;+\; \langle B^*{}_-(t)B^*{}_-(t)\rangle \tag{87}$$

Then on account of Equations. (58-60), we note that

$$\langle B^*{}_+(t)B^*{}_+(t)\,\rangle = 0. \tag{88}$$

Following similar procedure, we find

$$\langle B^*{}_-(t)B^*{}_-(t)\,\rangle = 0\;, \tag{89}$$

$$\langle B^*{}_+(t)B^*{}_-(t)\,\rangle = 0. \tag{90}$$

And

$$\langle B^*{}_-(t)B^*{}_+(t)\rangle \;= 0\,. \tag{91}$$

Hence substituting of Equations. (89-91) into Equation. (88) result in

$$\langle\alpha^{*2}\,\rangle = 0\,. \tag{92}$$

Moreover, applying Equation. (87) and its complex conjugate and assuming that the cavity modes is initially in a two mode vacuum state, we see that

$$\langle \alpha^*(t)\alpha(t)\rangle = \langle B^*{}_+(t)B^*{}_+(t)\rangle + \langle B^*{}_+(t)B^*{}_-(t)\rangle + \langle B^*{}_-(t)B^*{}_+(t)\rangle + \langle B^*{}_-(t)B^*{}_-(t)\rangle, \quad (93)$$

Hence on basis of Equations. (78), (79) and (86), we get

$$\langle B^*{}_+(t)B^*{}_+(t)\rangle = \frac{K\bar{n}-\varepsilon}{2}\int_0^t e^{-\frac{\zeta\pm}{2}(2t-t'-t'')}\delta(t'-t'')dt'dt'' . \quad (94)$$

Applying the properties of delta function and upon carrying out the integration over t′, we find

$$\langle B^*{}_+(t)B^*{}_+(t)\rangle = \frac{K\bar{n}-\varepsilon}{2}\left(1-e^{-\zeta_+}t\right). \quad (95)$$

Furthermore, it can also be established in a similar manner that

$$\langle B^*{}_-(t)B^*{}_-(t)\rangle = \frac{K\,\bar{n}-\varepsilon}{2}\left(1-e^{-\zeta_+}t\right), \quad (96)$$

$$\langle B^*{}_+(t)B^*{}_-(t)\rangle = \langle B^*{}_-(t)B^*{}_+(t)\rangle = 0 . \quad (97)$$

Substituting Equations. (95 – 97) into Equation. (93), we get

$$\langle \alpha^*(t)\alpha(t)\rangle = \frac{1}{2\zeta_+}(K\bar{n}-\varepsilon)\left(1-e^{-\zeta_+}t\right) + \frac{1}{2\zeta_+}(K\bar{n}+\varepsilon)\left(1-e^{-\zeta_-}t\right). \quad (98)$$

Following the same procedure, we readily obtain

$$\langle \beta^*(t)\beta(t)\rangle = \frac{1}{2\zeta_+}(K\hat{n}-\varepsilon)\left(1-e^{-\zeta_+}t\right) + \frac{1}{2\zeta_-}(K\bar{n}+\varepsilon)\left(1-e^{-\zeta_-}t\right), \quad (99)$$

$$\langle \alpha^*(t)\beta^*(t)\rangle = \frac{1}{2\zeta_+}(K\bar{n}-\varepsilon)\left(1-e^{-\zeta_+}t\right) - \frac{1}{2\zeta_-}(K\bar{n}+\varepsilon)\left(1-e^{-\zeta_-}t\right). \quad (100)$$

And

$$\langle \alpha(t)\beta(t)\rangle = \frac{1}{2\zeta_+}(K\bar{n}-\varepsilon)\left(1-e^{-\zeta_+}t\right) - \frac{1}{2\zeta_-}(K\bar{n}+\varepsilon)(1-e^{-\zeta_-}t). \quad (101)$$

Now upon substituting Equations. (98-101) into Equation. (85) leads to

$$\varphi_a(z,\eta,t) = \exp[-a(z^*z + \eta^*\eta) + b(z\eta + \eta^*z^*)] . \quad (102)$$

Finally, introducing Equation. (102) into Equation. (87), upon performing the integration and employing the relation

$$\int d^2\,y e^{-cy^*y+ay+by^*} = \frac{\pi}{c}e^{\frac{ab}{c}}. \quad (103)$$

The Q -function for the two mode sub harmonic generator coupled to thermal reservoir is found to

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}(u^2-v^2)\exp[-u(\alpha^*\alpha+\beta^*\beta) + v(\beta^*\alpha^*+\beta\alpha)] , \quad (104)$$

In which

$$u = \frac{a}{a^2-b^2} \;\; \text{and} v = \frac{b}{a^2-b^2}. \quad (105)$$

This is the Q- function for the sub-harmonic generator coupled to thermal reservoir.

D. The Density Operator

Here we seek to determine the density operator for two-mode light beams. Suppose $\hat{\boldsymbol{\rho}}(\hat{\boldsymbol{a}}^{+}, \hat{\boldsymbol{b}}^{+}, \boldsymbol{t})$ is the density operator for a certain two mode light beam. Then upon expanding this density operator in normal order [27]

$$\rho(\hat{a}^{+}, \hat{b}^{+}, t) = \sum_{p,q,r,s} Cpqrs\, \hat{a}^{+p}(t)\hat{b}^{+q}(t)\hat{a}^{r}(t)\hat{b}^{s}(t), \tag{106}$$

And recalling the completeness relation for a two-mode coherent-state

$$I = \frac{1}{\pi^2}\int d^2\alpha d^2\beta |\alpha, \beta\rangle\langle\beta, \alpha| . \tag{107}$$

On the other hand, the expectation value of an operator $\hat{\boldsymbol{A}}(\hat{\boldsymbol{a}}+, \hat{\boldsymbol{b}}+, t)$ can be expressed in the form of

$$\hat{A}(\hat{a}^{+}, \hat{b}^{+}, t) = \mathrm{Tr}\left(\hat{\rho}'(t)\hat{A}(0)\right). \tag{108}$$

To this end, applying the completeness relation given by Equation. (107) in (106) twice, we have

$$\rho(\hat{a}^{+}, \hat{b}^{+}, t) = \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\frac{d^2\eta}{\pi}\frac{d^2\lambda}{\pi}|\alpha, \beta\rangle\langle\beta, \alpha|\hat{\rho}\,|\eta, \lambda\rangle\langle\lambda, \eta| . \tag{109}$$

This can be rewritten as in the form

$$Q(\alpha^*, \beta^*, \eta, \lambda, t) = \frac{1}{\pi^2}\langle\beta, \alpha|\hat{\rho}\,|\eta, \lambda\rangle. \tag{110}$$

Therefore, in view of Equations. (107) and (110), the expectation value of a given operator function $\hat{\boldsymbol{A}}(\hat{\boldsymbol{a}}+, \hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}+, \hat{\boldsymbol{b}}, t)$ is expressible as [27]

$$\langle\hat{A}(\hat{a}, \hat{b}, \hat{c}^{+}, \hat{d}^{+}, t)\rangle = \frac{1}{\pi^2}\int d^2\alpha d^2\beta d^2\eta d^2\lambda Q(\alpha^*, \beta^*, \eta, \lambda, t) \exp[-\alpha^*\alpha - \beta^*\beta - \eta^*\eta - \lambda^*\lambda + \eta^*\alpha + \alpha^*\eta + \beta^*\lambda + \lambda^*\beta] A_N(\eta^*, \lambda^*, \alpha, \beta), \tag{111}$$

Where,

$$|\langle\beta, \alpha|\eta, \lambda\rangle|^2 . \tag{112}$$

With $A_N(\eta*, \lambda^*, \alpha, \beta)$ is the c-number function corresponding to $\hat{\boldsymbol{A}}(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}}+, \hat{\boldsymbol{d}}+, t)$ in the normal order.

3. Photon Statistics

A. The mean photon number

Here our investigation is to calculate the mean photon number of the signal-idler modes coupled to thermal reservoir. The mean photon number for the signal-idler modes in terms of density operator can be expressed as [16-20].

$$\langle\hat{c}^{+}(t)\hat{c}(t)\rangle = \mathrm{TR}\left(\rho(t)\hat{c}^{+}(0)\hat{c}(0)\right), \tag{113}$$

In which

$$\hat{c} = \hat{a} + \hat{b} \text{ and } \hat{c}^{+} = \hat{a}^{+} + \hat{b}^{+}. \tag{114}$$

Where $\hat{a}, \hat{b}$ and $\hat{c}$ are the annihilation operators for a light mode a, light mode b, and the two-mode, idler mode, and the signal-idler modes, respectively Employing Equations. (114) an (113) can be written as

$$\langle \hat{c}^+(t)\hat{c}(t) \rangle = \langle \hat{a}^+(t)\hat{a}(t)\rangle + \langle \hat{a}^+(t)\hat{b}(t)\rangle + \langle \hat{b}^+(t)\hat{a}(t)\rangle + \langle \hat{b}^+(t)\hat{b}(t)\rangle, \quad (115)$$

It then follows that

$$\langle \hat{a}^+(t)\hat{a}(t) \rangle = \frac{1}{\pi^4}(u^2 - v^2)\frac{d^2}{dxdy}\int d^2\alpha d^2\beta\, d^2\eta d^2\lambda$$
$$\exp[-(u\eta\alpha^* - u\lambda\beta^*) + (v\lambda\eta + v\beta^*\alpha^*)]$$
$$\exp[-\alpha^*\alpha - \beta^*\beta - \eta^*\eta - \lambda^*\lambda + \eta^*\alpha + \alpha^*\eta + \beta^*\lambda + \lambda^*\beta + x\alpha + y\eta^*]|x = y = 0\,. \quad (116)$$

With the aid of the identity described by Equation. (115) and upon performing the integration over λ along with Equation. (116), Equation. (115) yields

$$\langle \hat{a}^+(t)\hat{a}(t)\rangle = \frac{1}{\pi^4}(u^2 - v^2)\frac{d^2}{dxdy}\int d^2\,\alpha d^2\beta d^2\eta$$
$$\exp[-(\eta^*\eta + \eta^*(\alpha + y) + \eta(\alpha^* + v\beta - v\alpha^*)]$$
$$\exp[-(\alpha^*\alpha + x\alpha + v\alpha^*\beta^* - v\beta^*\beta)]|x = y = 0. \quad (117)$$

So that carrying out the integration over β and η, there follows

$$\langle \hat{a}^+(t)\hat{a}(t)\rangle = \frac{1}{\pi^4}(u^2 - v^2)\frac{d^2}{dxdy}\int d^2\alpha\,\exp(-\alpha^*\alpha)\left(\frac{u^2 - v^2}{u}\right)$$
$$+\alpha^*(uy + u^2y + v^2y) + x\alpha)|x = y = 0. \quad (118)$$

Using Equation. (118) and performing differentiation, by applying the condition, x = y = 0, we readily obtain

$$\langle \hat{a}^+(t)\hat{a}(t) \rangle = a - 1, \quad (119)$$

Similarly, following the same procedure, we note that

$$\langle \hat{b}^+(t)\hat{b}(t)\rangle = a - 1. \quad (120)$$

Then in view of Equations. (119), and (120), Equation. (115) turns out to be

$$\langle \hat{c}^+(t)\hat{c}(t) \rangle = 2(a - 1). \quad (121)$$

Upon substituting Equation. (114) into Equation. (121), we get

$$\langle \hat{c}^+(t)\hat{c}(t) \rangle = \frac{2}{2(K + 2\varepsilon)}(K\bar{n} - \varepsilon)\left(1 - e^{-\zeta_+}t\right) + \frac{2}{2(K - 2\varepsilon)}(K\bar{n} + \varepsilon)\left(1 - e^{-\zeta_-}t\right)$$
$$+\frac{2}{2(K+2\varepsilon)}(K\bar{n} - \varepsilon)\left(1 - e^{-\zeta_+}t\right) + \frac{2}{2(K-2\varepsilon)}(K\bar{n} + \varepsilon)\left(1 - e^{-\zeta_-}t\right). \quad (122)$$

The mean photon number of signal-idler modes at steady-state turns out to be,

$$\langle \hat{c}^+(t)\hat{c}(t)\rangle = \frac{2K^2\bar{n}}{K^2-4\varepsilon^2} + \frac{4\varepsilon^2}{K^2-4\varepsilon^2}\,. \quad (123)$$

This shows that the mean photon number of the system does not happen to be the sum of the mean photon-number of the signal-idler modes and the thermal light.

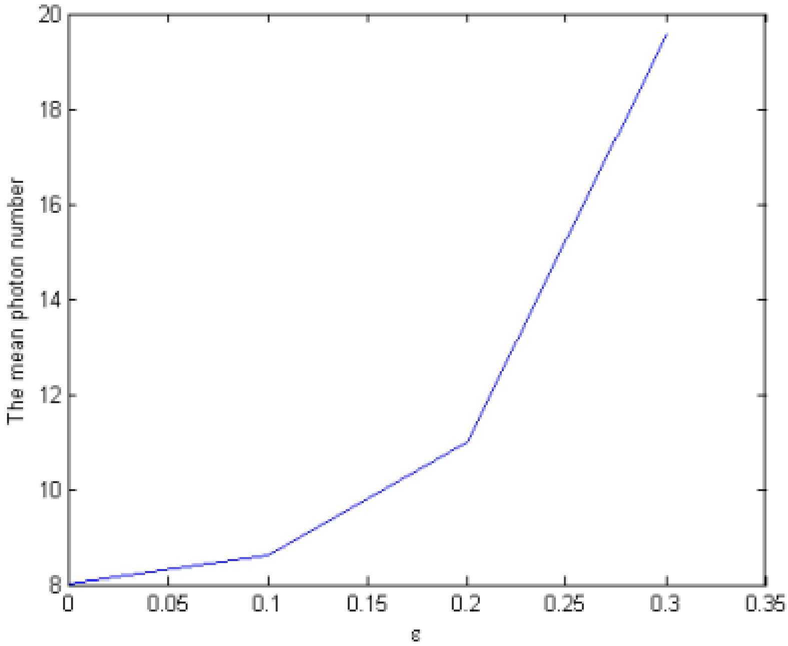

Figure 2. A plot of the mean photon number versus epsilon [Eq. 123] for K = 0:8 and $\bar{n}$ = 4

For condition in which $\bar{n}$ = 0, we see that

$$\langle \hat{c}^{+}(t)\hat{c}(t)\rangle = \frac{4\varepsilon^2}{K^2 - 4\varepsilon^2} . \qquad (124)$$

Is the mean photon number of the signal-idler modes coupled to vacuum reservoir.

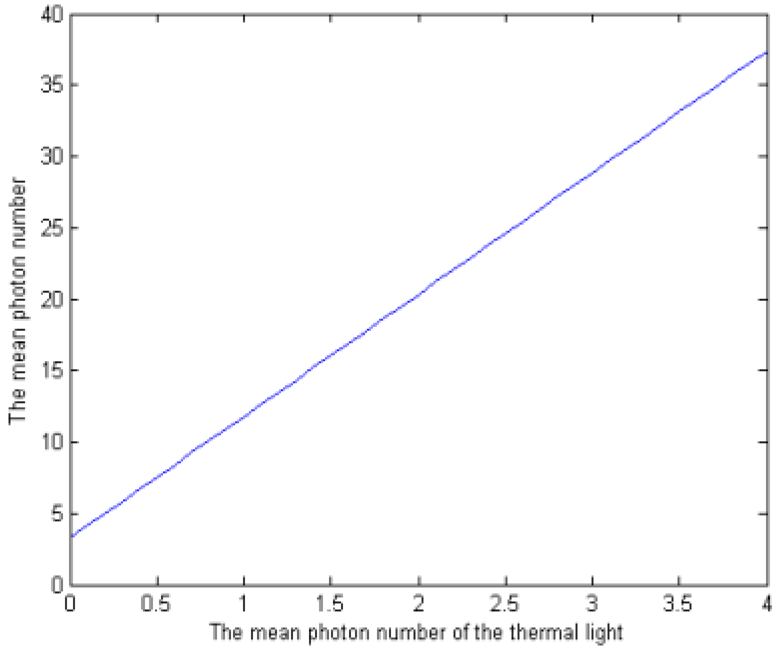

Figure 3. A plot of the mean photon number versus the mean photon number of thermal light [Eq. 124] for K = 0:8 and $\bar{n}$ = 0.

We immediately observe from Figure 2 as well as Figure 3, the mean photon number of the system increases with increasing ε and $\bar{n}$, respectively. Also Figure 3, shows that the mean photon number of the system increases a point above the origin with increasing ε and $\bar{n}$.

B. The variance of the photon number

We next proceed to determine the variance of the photon number for signal idler modes. Then we define the photon-number variance for signal-idler modes by [21-27].

$$(\Delta n)^2 = \langle(\hat{c}^+(t)\hat{c}(t))^2\rangle - \langle\hat{c}^+(t)\hat{c}(t)\rangle^2. \tag{125}$$

The quadrature operators $\hat{c}$ and $\hat{c}^+$ are Hermitian and satisfy the commutation relation

$$[\hat{c}, \hat{c}^+] = 2\,. \tag{126}$$

Hence employing Equation. (126), Equation. (125) becomes

$$(\Delta n)^2 = \langle\hat{c}^{+2}(t)\hat{c}(t)\rangle + 2\langle\hat{c}^+(t)\hat{c}(t)\rangle - \langle c^+(t)\hat{c}(t)\rangle\,, \tag{127}$$

We note that $\hat{c}(t)$ is a Gaussian operator with zero mean. Hence we see that

$$\langle\hat{c}^{+2}(t)\hat{c}^2(t)\rangle = 2\langle\hat{c}^+(t)\hat{c}(t))^2\rangle + \langle\hat{c}^{+2}(t)\rangle\langle\hat{c}^2(t)\rangle\,. \tag{128}$$

In view of the fact that the $\hat{a}(t)$ and $\hat{b}(t)$ are a Gaussian variable with zero mean, we see that

$$\langle\hat{a}\rangle = \langle\hat{a}^+\rangle = \langle\hat{b}\rangle = \langle\hat{b}^+\rangle = 0. \tag{129}$$

Now we calculate

$$\langle \hat{a}^{+}(t)\hat{b}^{2}(t)\rangle = \frac{1}{\pi^{4}}(u^{2}-v^{2})\int d^{2}\alpha d^{2}\beta d^{2}\eta d^{2}\lambda \exp[-(u\eta\alpha^{*}-u\lambda\beta^{*})+(v\lambda\eta+ v\beta^{*}\alpha^{*})\exp[-\alpha^{*}\alpha-\beta^{*}\beta-\eta^{*}\eta-\lambda^{*}\lambda+\eta^{*}\alpha+\alpha^{*}\eta+\beta^{*}\lambda+\lambda^{*}\beta]\alpha\beta \tag{130}$$

Hence performing the integration over λ, we get

$$\langle \hat{a}^{+}(t)\hat{b}^{+}(t)\rangle = \frac{1}{\pi^{3}}(u^{2}-v^{2})\frac{d^{2}}{dpdq}\int d^{2}\alpha d^{2}\beta d^{2}\eta \exp\left[-\left(\eta^{*}\eta+\eta^{*}\alpha^{*}+\eta(\alpha^{*}+v\beta- v\alpha^{*})\right)\right]\exp[-(\alpha^{*}\alpha+p\alpha+q\beta+v\alpha^{*}\beta^{*}-v\beta^{*}\beta)]|p=q=0 \; . \tag{131}$$

Upon carrying out the integration over α, β, and η using the identity in Equation. (131), we find

$$\langle \hat{a}^{+}(t)(t)\hat{b}^{+}\rangle(t)\rangle = \frac{d^{2}}{dpdq}\left(\frac{vpq}{e^{u^{2}-v^{2}}}\right)|p=q=0. \tag{132}$$

So that performing the differentiation and applying the condition p=q=0, one easily obtains

$$\langle \hat{a}^{+}(t)\hat{b}^{+}(t)\rangle = \frac{v}{u^{2}-v^{2}} = b \; , \tag{133}$$

Following a similar procedure, we readily find

$$\langle \hat{a}(t)\hat{b}^{+}(t)\rangle = \frac{v}{u^{2}-v^{2}} = b\,. \tag{134}$$

Introducing Equations. (148) and (149) in Equation. (144), we have

$$(\Delta n)^{2} = -4a+4a^{2}+4b^{2}\,. \tag{135}$$

Upon substituting Equations. (122) and (123) into Equation. (135), the photon number variance of signal-idler modes at steady state turns out to be

$$(\Delta)^{2} = \frac{4K^{2}\bar{n}}{K^{2}-4\varepsilon^{2}}+\frac{8\varepsilon^{2}}{K^{2}-4\varepsilon^{2}}+\frac{4K^{4}\bar{n}^{2}}{(K^{2}-4\varepsilon^{2})(K^{2}-4\varepsilon^{2})}+\frac{32K^{2}\bar{n}^{2}\varepsilon^{2}}{(K^{2}-4\varepsilon^{2})(K^{2}-4\varepsilon^{2})} +\frac{16\varepsilon^{4}}{(K^{2}-4\varepsilon^{2})(K^{2}-4\varepsilon^{2})}+\frac{16K^{2}\bar{n}^{2}\varepsilon^{2}}{(K^{2}-4\varepsilon^{2})(K^{2}-4\varepsilon^{2})}+\frac{4K^{2}\varepsilon^{2}}{(K^{2}-4\varepsilon^{2})(K^{2}-4\varepsilon^{2})}\,. \tag{136}$$

C. The photon number distribution

We wish to obtain an explicit expression for the photon number distribution by using the density operator of signal-idler modes coupled to thermal reservoir, one can write

$$P(n,m,t) = \langle n,m|\rho(\hat{a}^{+},\hat{b}^{+},t)|m,n\rangle\,. \tag{137}$$

It then follows that

$$P(n,m,t) = \frac{1}{\pi^{4}}(u^{2}-v^{2})\int d^{2}\alpha d^{2}\beta d^{2}\eta d^{2}\lambda \exp[-(u\eta\alpha^{*}-u\lambda\beta^{*}+v\lambda\eta+v\beta^{*}\alpha^{*})]\exp(\frac{\alpha^{*}\alpha}{2}- \frac{\beta^{*}\beta}{2}-\frac{\eta^{*}\eta}{2}-\frac{\lambda^{*}\lambda}{2}(\lambda^{*}\beta)^{m}(\eta^{*}\alpha)^{n}\;)\,. \tag{138}$$

Where,

$$\langle m,n|\alpha,\beta\rangle = \exp\left(-\frac{\alpha^{*}\alpha}{2}-\frac{\beta^{*}\beta}{2}\left(\frac{\alpha^{n}\beta^{m}}{\sqrt{n!m!}}\right)\right). \tag{139}$$

Upon carrying out the integration over α, β, η and λ, we readily obtain

$$P(n,m,t) = (u^2 - v^2)\frac{d^{2m}}{d_z{}^m d_z{}^{*m}}\frac{d^{2n}}{d_\gamma{}^n d_\gamma{}^{*n}}(4v\gamma z + 4u\gamma\gamma^* - 4uz^*z - 4vz^*\gamma^*)$$
$$|z = z^* = \gamma = \gamma^* = 0\,. \tag{140}$$

Upon performing the differentiation and applying the condition z = z∗ = γ = γ∗ = 0, we find

$$P(n,m,t)$$
$$= (u^2 - v^2)16\sum_{i,j,k,l}\frac{4^{i+j+k+l}u^{i+j}v^{l+k}}{i!\,j!\,k!\,l!}\frac{(i+l)!}{(i+l-m)!}\frac{(i+k)!}{(i+k-m)!}\frac{(j+l)!}{(j+l-n)!}\frac{(j+k)!}{(j+k-n)!}\delta i$$
$$+l\delta i + k\delta j + l\delta j + k. \tag{141}$$

We note that k=l= m-i=n-j. Therefore, for m=n the photon number distribution takes the form

$$P(n,m,t) = u^2 - v^{2\sum_{i=0}^{n}}\frac{n!^2 2^{2(2+n)}(-1)^n v^{2(n-i)}u^{2i}}{i!^2[(n-i)!]^2}. \tag{142}$$

Where n is the even number of photons in the cavity. From this result, we observe that the probability to observe of n signal photons and n idler photons inside the cavity.

4. Quadrature Fluctuations

A. Quadrature variance

We now proceed to determine the variance of the quadrature operators for the signal mode produced by a two mode sub harmonic generator coupled to thermal reservoir. The squeezing properties of a two-mode light can be described by two quadrature operators defined by [2].

$$\hat{c}_+ = \hat{c}^+ + \hat{c}\ , \tag{143}$$

And

$$\hat{c}\_ = i(\hat{c}^+ - \hat{c})\ . \tag{144}$$

In which $\hat{c}$+ and $\hat{c}$_are the plus and minus quadrature operators and with the aid of Equations. (128), (143), and (144), one can easily verify the commutation relation

$$[\hat{c}^+, \hat{c}\_] = 4i. \tag{145}$$

Then on the basis of Equations. (143) and (144), we readily obtain

$$(\Delta\hat{c}_\pm)^2 = 2 \pm \langle\hat{c}^{+2}\rangle + 2\langle\hat{c}^+\hat{c}\rangle \pm \langle\hat{c}^2\rangle \mp \langle\hat{c}^{+2}\rangle - 2\langle\hat{c}^+\hat{c}\rangle \mp \langle\hat{c}^2\rangle\,, \tag{146}$$

Since $\hat{a}$ and $\hat{b}$ are Gaussian variables with zero mean. Then $\hat{c}$ is also a Gaussian variable with zero mean. Hence we have

$$(\Delta\hat{c}_\pm)^2 = 2 \pm \langle\hat{c}^{+2}\rangle + 2\langle\hat{c}^+\hat{c}\rangle \pm \langle\hat{c}^2\rangle\,. \tag{147}$$

Thus on account of Equations. (133) and (144), Equation. (147) turns out to be

$$(\Delta\hat{c}_\pm)^2 = 2 + 2\langle\hat{a}^+(t)\hat{a}(t)\rangle + 2\langle\hat{b}^+(t)\hat{b}(t)\rangle \pm 2\langle\hat{a}^+(t)\hat{b}^+(t)\rangle \pm 2\langle\hat{a}(t)\hat{b}(t)\rangle. \tag{148}$$

So that c-number variables corresponding to Equation. (148) is

$$(\Delta\hat{c}_\pm)^2 = 2 + 2\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle \pm 2\langle\alpha^*(t)\beta^*(t)\rangle \pm 2\langle\alpha(t)\beta(t)\rangle\,. \tag{149}$$

At steady state, the quadrature variance of signal modes is

$$(\Delta\hat{c}_{\pm})^2 = 2 + 4\frac{(K\bar{n} \mp \varepsilon)}{K \pm 2\varepsilon}, \tag{150}$$

Upon setting $\bar{n} = 0$ we see that

$$(\Delta\hat{c}_{\pm})^2 = 2 + \frac{\mp 4\varepsilon}{K \pm 2\varepsilon}. \tag{151}$$

B. Quadrature squeezing

We wish to calculate the squeezing of the two mode sub harmonic generator relative to the variance of the two mode sub-harmonic generator. We therefore defined by the quadrature squeezing of the two mode sub harmonic generator by [2]

$$S = \frac{2-(\Delta\hat{c}_{\pm})^2 \text{ system}}{2}. \tag{152}$$

Where, S shows the quadrature squeezing of the two mode sub harmonic generator so that on account of Equation. (152) there follows

$$S_{+} = -2\frac{(K\bar{n}-\varepsilon)}{K+2\varepsilon}\left(1 - e^{-\zeta_{+}t}\right), \tag{153}$$

And at steady state takes the form

$$S_{+} = -2\frac{(K\bar{n}-\varepsilon)}{K+2\varepsilon}. \tag{154}$$

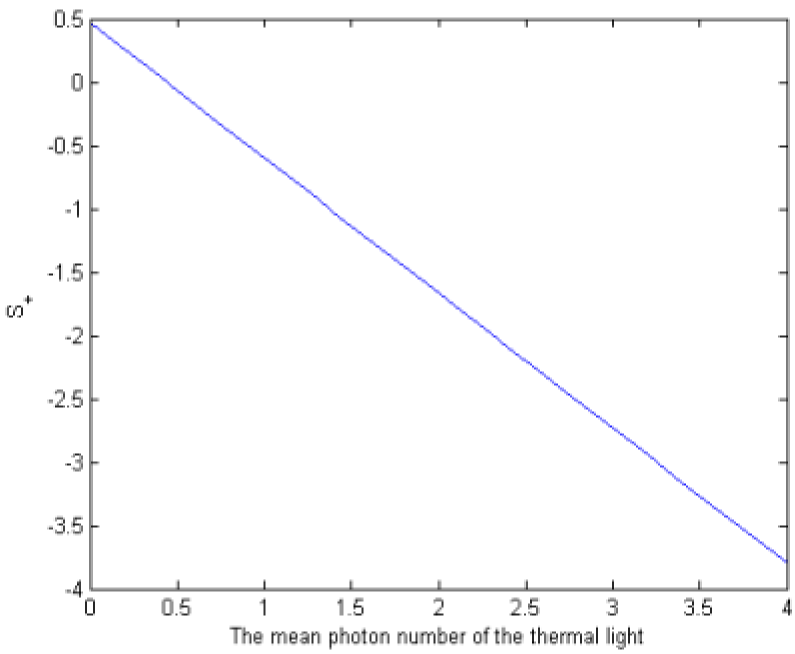

Figure 4. A plot of Quadrature squeezing versus the mean photon of the thermal light [Eq. 154] for K=0.8 and $\boldsymbol{\varepsilon}$ = 35

Moreover on taking into account Equation. (154), we see that at threshold

$$S_{+} = \frac{2\varepsilon}{K + 2\varepsilon}\ . \tag{155}$$

Now at threshold, we observe that there is 40% squeezing of the output light and 50% squeezing of the cavity light below the vacuum level when single-mode sub harmonic generator is coupled to thermal reservoir.

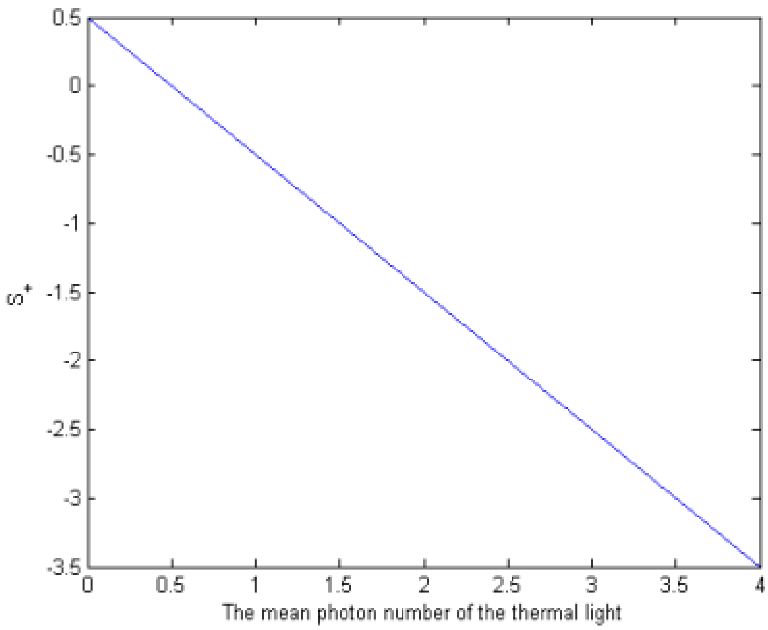

Figure 5. A plot of Quadrature squeezing versus the mean photon of the thermal light [Eq.155].

We observe that the signal-idler modes are in a squeezed state and the squeezing occurs in the plus quadrature. From the plot in Figure 4 and Figure 5, we see that the degree of squeezing is indeed affected by the present of thermal light.

5. Conclusion

In this article we have studied the squeezing and the statistical properties of the light, produced by a two mode sub harmonic generator coupled to thermal reservoir. We have first obtained the master equation and the differential equations. Employing these equations, we have obtained the solutions of c-number Langavin equations. Applying these solutions of c-number Langavin equations along with the anti-normally ordered characteristic function, we have calculated the Q function. With the aid of this Q function, we have calculated the mean and the variance of the photon number of signal-idler modes coupled to thermal reservoir. Furthermore, the density operators in terms of the Q function is then used to calculate the photon number distribution. We have calculated the quadrature variance and quadrature squeezing. Finally, we have found that the degree of squeezing is indeed affected by the present of thermal light. However, the mean photon number of the system under consideration increases with increasing $\bar{n}$.

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