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Communication

Spin Structures and non-Relativistic Spin Operators

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Abstract: In Quantum Physics, the spin and angular momentum operators are magnitudes introduced by means of a vector transformation law. However, interpreting the eigenvalues of its Z "components" as projections on said axis leads to certain contradictions supposedly avoided by a mandatory (presented as a freely selected) Z's orientation. It is shown that an oriented physical space almost forces us to project the angular momentum's and spin's eigenvalues onto its orientation's 3-form, which sidesteps entering into inconsistencies. The final conclusion is that this "rare" magnitude called spin, downright naturally comes in and plays thanks to the orientation of our three-dimensional space.

Keywords: Irreducible representations, tangent bundle, angular momentum, spin, 2-covering, spin structure, manifold orientation, left-handed bias

1. Introduction

The consideration of the angular momentum operators does not generate many teething troubles, event its eigenvalues are not exactly geometrical projections according to the well-known Elementary Geometry meaning. The cause is, that these ones are perceived through a cross-product definition, where in a natural way appear the Cartesian coordinates x, y and z. Nevertheless, the situation with respect to the spin introduction is very different – more concretely, the infinitesimal generators of the presentations into the bi-dimensional space $\mathbb{R}^2$ (i.e. the Pauli matrices), cannot be associated naturally with the current coordinates. One says in the physical space, it is selected a random direction (traditionally coincided with the axel Z), onto which it is projected the Z spin component understanding under this the $\hat{s}_z$ eigenvalues. But one can have a conflict since, if a second direction (not coincident with the first) is chosen at random in order to project the same spin, the values of the two projections could be quite different. We therefore need an axis onto which the $\hat{s}_z$ eigenvalues considered as Z "projections" do not change under any one $SO(3)$ transformations. In relativistic physics, although from a different approach, a similar problem arises. In this article, we will prove that such axis can be the bases axis (of the unidimensional space) generated by a beforehand fixed space orientation in $\mathbb{R}^3$, namely the differential 3-forme:

$$dx \wedge dy \wedge dz$$

which sets the $\mathbb{R}^3$space orientation (considered already this space as a manifold).
On the other hand, the mathematicians define the existence of a spin structure for $T(\mathbb{R}^3)$ (see, for example, Kirby [1]) through the following (adapted here for our particular case) commutative diagram:

$$\tilde{T}(\mathbb{R}^3) \times SU(2) \xrightarrow{SU(2)\ action} \tilde{T}(\mathbb{R}^3)$$

$$\varpi \times \pi \downarrow \qquad\qquad \downarrow \varpi$$

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$$T(\mathbb{R}^3) \times SO(3) \xrightarrow{SO(3)\ action} T(\mathbb{R}^3)$$

It is required that the $SU(2) \equiv Spin(3)$ on the right actions are realized faithfully and transitively. Here, ϖ and π are the projections onto the tangent bundle $T(\mathbb{R}^3)$ and $SO(3)$ from its respective 2-fold coverings: $\tilde{T}(\mathbb{R}^3)$ and the $SU(2)$. A similar procedure could be carried out not only for flat spaces, but also in cases of fiber bundles above certain curved base spaces (as the usually "inheriting" ones from the General Relativity Theory [2]). The point is that, roughly speaking, the $SU(2)$ acts over the complexified manifold $\tilde{T}(\mathbb{R}^3)$ and this one has a natural orientation that is reproduced indirectly onto $T(\mathbb{R}^3)$ through the π projection (φ_1, θ and φ_2 are the three standard Euler rotation angles introduced counter-clockwise presenting old coordinates by the new ones – let us note that the relationship old ↔ new although implicitly means a temporal ordering, something that we will use below) as follows from the explicit description of the $SU(2)$ and $SO(3)$ *group* elements [2] (the signs ∓ correspond to the different covering folds):

$$\mp\begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\varphi_2+\varphi_1}{2}} & i\sin\frac{\theta}{2}e^{i\frac{\varphi_2-\varphi_1}{2}} \\ i\sin\frac{\theta}{2}e^{-i\frac{\varphi_2-\varphi_1}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\varphi_2+\varphi_1}{2}} \end{pmatrix}$$

$$\Big|\ \pi\Big|\ \downarrow$$

$$\begin{pmatrix} \cos\varphi_2\cos\varphi_1 - \cos\theta\sin\varphi_2\sin\varphi_1 & -\cos\varphi_2\sin\varphi_1 - \cos\theta\sin\varphi_2\cos\varphi_1 & \sin\theta\sin\varphi_2 \\ \sin\varphi_2\cos\varphi_1 + \cos\theta\cos\varphi_2\sin\varphi_1 & -\sin\varphi_2\sin\varphi_1 + \cos\theta\cos\varphi_2\cos\varphi_1 & -\sin\theta\cos\varphi_2 \\ \sin\theta\sin\varphi_1 & \sin\theta\cos\varphi_1 & \cos\theta \end{pmatrix}$$

Nevertheless, being $T(\mathbb{R}^3)$ (with $\mathbb{R}^3$) connected and single connected, its 2-fold covering must be a non-connected one (see Narasimhan [3]). From the over said, it is easy to deduct (doing it fibre by fibre) that $\tilde{T}(\mathbb{R}^3) \cong -\widehat{T(\mathbb{R}^3)} \cup \widehat{T(\mathbb{R}^3)}$ (where the corresponding components to the ∓ signs denote the respective orientations) and, they are disjoint: $-\widehat{T(\mathbb{R}^3)} \cap \widehat{T(\mathbb{R}^3)} = \emptyset$. At this point we are forced to choose one of the two oriented manifolds $\widehat{\mathbb{R}^3}$ or $-\widehat{\mathbb{R}^3}$ as a model for our physical space, since there is no human communication between them. Let us select $\widehat{\mathbb{R}^3}$. The space $\mathbb{R}^3$ with the opposite one of the chosen just beforehand orientation keeps the same cause-consequence ordering, which is implicitly encoded in the relationship old ↔ new coordinates (we have used, that the flat space orientations naturally are carried onto its tangent spaces and vice versa, therefore $\widehat{T(\mathbb{R}^3)} = T(\widehat{\mathbb{R}^3})$ and $-\widehat{T(\mathbb{R}^3)} = T(-\widehat{\mathbb{R}^3})$; furthermore, the used isomorphisms are not explicitly explained).

The second chapter shows why and how the eigenvalues of the quantum angular momentum are projected onto the orientation's 3-form of $\mathbb{R}^3$, while the third chapter is dedicated to the spin operator introduction. At the end, certain conclusions on the everywhere (beginning from the elementary particles Physics [4], going through the snails' anatomy [5] and reaching the galaxy structures [6]) observed left-handed bias phenomena and its implications are drawn.

2. Eigenvalues' projections of the quantum angular momentum

Conventionally, the quantum angular momentum $\hat{l}$ is considered as an operator-vector, whose mean values of its components transform as the components of an ordinary $SO(3)$ vector, while the mean value of its square is a scalar. The above affirmations are equivalent to the following well-known transformation laws [7] (the sum by j is implicit):

$$\hat{R}_{g}\hat{l}_i\hat{R}_{g}^{-1} = g_i^j\,\hat{l}'_j$$

$$\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2 = \hat{l}'^2_x + \hat{l}'^2_y + \hat{l}'^2_z = \hat{l}'^2$$

i, j go through x, y and z; the operators $\hat{R}_{g}$ are representation of the matrices $g \in SO(3)$ into the separable Hilbert space H_s (we mean the existent isomorphism among all such spaces) traditionally considered as broken down by the subspaces of 1, 3, 5, ... , and etc. dimensions where, the respective $SO(3)$ irreducible linear representations are unique (with equivalence exactitude). However, an important question remains: how are, for example, the $\hat{l}_z$ eigenvalues transformed?

It is known, that the independent common $\hat{l}^2$ and $\hat{l}_z$ eigenvectors, corresponding to each eigenvalue $l = 0, 1, 2, \dots$, form bases of the mentioned finite spaces while, the $\hat{l}_z$ eigenvalues enumerate the basic vectors, i.e. the spherical functions Y_{ll_z}, $l_z = -l, -l + 1, \dots, 0, \dots, l-1, l$. It is also clear that, if the coordinate system where the arguments of the spherical functions are registered is $SO(3)$ changed, their subscripts (i.e. l_z eigenvalues) remain unchanged, speaking at the same time about the " Z projections" of the operator $\hat{l}_z$. Although the definition of $\hat{l}_z$ through the replacement of the classical magnitudes by the respective quantum operators is completely understandable, it is also true that, the following Descartes coordinates' expression (coming from the well-known formula $\hat{l} = \hat{r} \times \hat{p}$, [8,9] for example)

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

describes the rotational increments around the Z axis, the concern is that the eigenvalues l_z are not at all, the commonly known geometry projections by Z.

In the introduction, the orientability of our physical 3-dimensional space was modelled without taking into account the General Relativity. There is a substantial difference between an orientable space and an oriented one. In the second case, $\mathbb{R}^3$ always has to be considered together with its orientation i.e. with the form $dx \wedge dy \wedge dz$ (or with benchmark repers). This means that $\mathbb{R}^3$ always must be considered as a differentiable manifold (in order to be endowed with tangent or cotangent fibre bundles). Even if one such manifold is not compact, the Hilbert space elements are functions, whose improper integrals can be meaningful. But, more interesting is the question: what differs an oriented physical space from one without orientation? In an oriented space must exist some phenomena which to indicate certain preferences to right or to left turns. There exist and it is the left-handed bias: according to the below construction (possibly qualitative), it can be observed, almost in all the naturally created structures.

In this order of ideas, to calculate here and now whatever $\hat{l}_z$'s average value, we must evaluate the following integral [10]:

$$\int_{\mathbb{R}^3_{manifold}} \omega$$

from the 3-form $\omega = \psi^* \hat{l}_z \psi dx \wedge dy \wedge dz$; here ψ is whatever normalized quantum state's function. If the state function ψ is not an eigenvector of the operator $\hat{l}_z$, then this formalism does not lead to something new – the components transformation's formula only will be rewritten in new terms. But, if ψ_{l_z} is an eigenvector, we obtain that $\omega = l_z |\psi_{l_z}|^2 dx \wedge dy \wedge dz$ which means: the respective eigenvalue l_z, strictly speaking, must be "projected" onto the (differential) 3-form $dx \wedge dy \wedge dz$ over $\mathbb{R}^3$, where it belongs. It is very easy to verify that, with the same l_z's value the form ω can be reconstructed as is, in any other according $SO(3)$ transformed system. In this case, as a true projection onto the orientation can be consider the density Z component of the angular momentum.

3. A less traditional introduction of the spin operator

With the traditional spin introduction, the situation becomes even more delicate. In this case, over the years [8,9,11], it is said that randomly an axis is chosen (but, it is commonly the Z axis coincident with a magnetic or electric, etc. fields); and onto it are projected the spin (eigen)values. However, a paradox arises. Let us suppose, that initially we have chosen the Z axis along (or perpendicularly to) a constant magnetic field onto which we mark a spin non-zero projection. If we make a rotation in such a way, that the X axis replaces Z, and then proceed with respect to X in the same way as for Z, we must have projection onto Z ($\perp X$) equal to 0. So, the axis selection is not random. Subsequently, we will propose another point of view on the spin.

Our starting point is to involve the rest of the $SU(2)$ irreducible finite-dimensional representations, via $SO(3)$ and corresponding to $\widehat{\mathbb{R}^3}$ i.e. when the representation's identification number j is semi-integer [7]. The last symbol indicates whichever of the Casimir operator (i.e. $\hat{J}^2$) eigenvalues: $1/2$, $3/2$, $5/2$, ..., etc. Similarly, as it was done in the previous chapter, we can unite these finite spaces respectively of $2, 4, 6, \ldots$, etc. dimensions in a separable Hilbert space $H_s^{\updownarrow}$ consisting of the columns

$$\begin{pmatrix} \psi^{\uparrow} \\ \psi^{\downarrow} \end{pmatrix} = \psi^{\uparrow}\chi_{1/2} + \psi^{\downarrow}\chi_{-1/2}$$

with its well-known interior product definition. Due to the isomorphism between H_s, and $H_s^{\updownarrow}$ we will keep the same symbol for the operator $\hat{l}$, which now acts onto the components of the mentioned columns. In this order of ideas, it is curious to analyse the following operator difference (both $\hat{J}$ and $\hat{l}$ act in $H_s^{\updownarrow}$) called spin-vector:

$$\hat{s} = \hat{J} - \hat{l}$$

Let the $\hat{J}$'s degenerate eigenvalues be fixed by the pair J, J_z . Then $J_z = -J, -J+1, \ldots, J-1, J$ where J is the eigenvalue of the operator $\hat{J}^2$ and J_z of $\hat{J}_z$. Usually, the eigenvectors of the last operators are presented by 2-row column matrices just like that (note, mathematically the $\hat{J}^2$ eigenvalues are $J(J+1)$, the same for $\hat{l}^2$and $\hat{s}^2$):

$$\begin{pmatrix} \Upsilon_{l;Jl+1/2} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \Upsilon_{l;Jl-1/2} \end{pmatrix}$$

As a result, the Hilbert space H_s is "restructured" in $H_s^{\updownarrow}$. The relationship between the basic functions Y_{ll_z} and the above symbols $\Upsilon_{l;JJ_z}$ is done by the Clebsch-Gordan coefficients $\langle s_z l_z | l, JJ_z \rangle$ [7,12] (the subscript $1/2$ corresponding to $\hat{s}^2$ is omitted):

$$\Upsilon_{l;JJ_z}(\vartheta, \phi) = \sum_{l_z=-l}^{l} \sum_{s_z=-\frac{1}{2}}^{\frac{1}{2}} \langle s_z l_z | l; JJ_z \rangle Y_{ll_z}(\vartheta, \phi)\, \chi_{s_z}(J_z)$$

Consequently, this way of introduction of the spin operator $\hat{S}$, allows us through the formula $\hat{s}_z = \hat{J}_z - \hat{l}_z$ to relate the non-relativistic spin to a reference frame. Obviously, the $\hat{J}_z$ eigenvalues are "projected" onto the 3-form $dx \wedge dy \wedge dz$ basis vector too.

Now we are ready to give some ideas on how to get to the left-handed bias from the orientation of our physical space. We consider that the energy in each specific case has a lower limit, if not, we would obtain an inexhaustible reserve of it (see, for example [13]), releasing the system bound states. On the other hand, due to its kinematic component, the same energy can grow up to infinity. Thus, if the unpaired spins or total moments of our hypothetical system's elements contribute opposite amounts of energy $\psi^{*}\hat{H}\psi dx \wedge dy \wedge dz$ according to their orientations, then the sum projection on the axis $dx \wedge dy \wedge dz$ will be privileged in one direction. The well-known correlation between orbital moments, spins and magnetic moments would suggest how the left-handed bias appear. It is valid for positive, but different energy inputs.

4. Discussions

As a whole, it is easy to realize, the above considerations about the spin magnitude's introduction have four important aspects: the first one takes in account the scientific rigor and this allows to place the non-relativistic spin concept in its place, carrying out all the calculations according to the appropriate mathematical formalism; on the other hand, the above opens an opportunity to explain the left-handed bias; the third aspect is to handle the spin correlations as certain calculus on manifolds, where by now, the differential forms get their values into the ring of the 2×2 matrices; the fourth aspect has a didactic value – from a didactic point of view, it is (very) disconcerting that the traditional methods of introducing spin generate certain doubts in students due to the fact that the teacher tries to locate a “dimension” in the space $\mathbb{R}^3$, without this belonging to it from a logical-formal point of view while, we really often “project” the scalars $\hat{l}^2$, $\hat{j}^2$ and $\hat{s}^2$ onto the axis Z too. This shows that mathematically spiking it is not about projections onto axes.

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